# Solutions to HW 10

EML 4312

**Problem 1:**

$$\left.\begin{aligned} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= -xz + rx - y \\ \dot{z} &= xy - bz \end{aligned}\right\} \quad (1)$$

It is not a linear system.

Informally, it is non-linear since $xz$ and $xy$ terms are non-linear functions of states $x, y, z$.

Formally, it is not linear because it cannot be expresed as $\dot{\mathbb{X}} = \mathbb{A}\mathbb{X}$ (there is no $\mathbb{U}$, so $\mathbb{B}\mathbb{U}$ is not needed)

If you try, you'll get something like

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \underbrace{\begin{bmatrix} -\sigma & \sigma & 0 \\ -z+r & -1 & 0 \\ y & 0 & -b \end{bmatrix}}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\mathbb{A}$ : is a function of the state $z, y$,

whereas for a linear system $\mathbb{A}$ is required to be ~~either a matrix of constants or a matrix of~~ either a constant matrix or a matrix that is an explicit function of time.

2. $\frac{Y}{U} = \frac{1}{s^2+5s+26} = G(s)$

$\Rightarrow (s^2+5s+26)Y(s) = U(s)$

$\Rightarrow \ddot{y} + 5\dot{y} + 26y = u$ (by taking inverse Laplace transform)

Define $x_1 = y$

$x_2 = \dot{x}_1 \; (= \dot{y})$

$\Rightarrow \dot{x}_1 = x_2$

$\dot{x}_2 = \ddot{y} = u - 5\dot{y} - 26y = u - 5x_2 - 26x_1$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -26 & -5 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

eigenvalues of A: roots of $\det(\lambda I - A) = 0$

$$\Leftrightarrow \det\begin{pmatrix} \lambda & -1 \\ 26 & \lambda+5 \end{pmatrix} = 0$$

$$\Leftrightarrow \lambda^2 + \lambda 5 + 26 = 0 \quad \text{—— (2)}$$

$$\Leftrightarrow \lambda = \frac{-5 \pm \sqrt{25-104}}{2} = \frac{-5 \pm \sqrt{79}\,j}{2}$$

comparing (2) with the denominator of $G(s)$, we see that eigenvalues of A are precisely the poles of $G(s)$.

[It is no surprise that they are equal, since we have proved in class that they are.]

3/ $\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -b & -k \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$

To check (complete) controllability of $(A, B)$, evaluate

$$\mathbb{P}_c = [B \quad AB] = \begin{bmatrix} 0 & 1 \\ 1 & -k \end{bmatrix}$$

To check if the rank of $\mathbb{P}_c$ is 2, evaluate

$\det(\mathbb{P}_c) = 0 - 1 = -1 \neq 0$

$\Rightarrow \text{rank}(\mathbb{P}_c) = 2 \Rightarrow (A, B)$ is completely controllable no matter what $b$ and $k$ are.

4/ $\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -.98 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 10.78 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.1 \\ 0 \\ -0.1 \end{bmatrix} u$ (this is the cart-n-pendulum system)

$n = 4$.

$$\mathbb{P}_c = [B \quad AB \quad A^2B \quad A^3B]$$

$$= \begin{bmatrix} 0 & 0.1 & 0 & 0.098 \\ 0.1 & 0 & 0.098 & 0 \\ 0 & -0.1 & 0 & -1.078 \\ -0.1 & 0 & -1.078 & 0 \end{bmatrix}$$

$\text{rank}(\mathbb{P}_c) = 4$ since $\det(\mathbb{P}_c) = 0.0096$ $(\neq 0)$

$\Rightarrow$ completely controllable.

Problem 5 : A in both cases is the same: $\begin{bmatrix} 0 & 1 \\ -b & -k \end{bmatrix}$

case 1 $\quad C = [1 \quad 0]$

$\Rightarrow \mathbb{P}_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ : rank 2 $\Rightarrow$ completely reconstructible

case 2 $\quad C = [0 \quad 1]$

$\Rightarrow \mathbb{P}_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -k \end{bmatrix}$ $\quad \det(\mathbb{P}_o) = b \begin{cases} \neq 0 & \text{if } b \neq 0 \\ = 0 & \text{if } b = 0 \end{cases}$

$\therefore$ rank $(\mathbb{P}_o) = 2$ if and only if $b \neq 0$

$\Rightarrow (A, C)$ is reconstructible if $b \neq 0$, otherwise not.

Problem 6 One option is to compute the transfer function by directly applying the formula

$$G(s) = C(sI - A)^{-1}B$$

But that requires inversion of a $3 \times 3$ matrix.

The simpler method is to take Laplace transform of each of the state equations and eliminate all the states except $x_1(s)$, (which is $y(s)$)

We have the state equations as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2x_1 - x_2 + u \qquad (\text{and} \quad y = x_1)$$

$$\dot{x}_3 = a x_3 ,$$

Taking Laplace transfom, we get

$$\left.\begin{array}{l} sX_1(s) = X_2(s) \\ sX_2(s) = -2X_1(s) - X_2(s) + U(s) \end{array}\right] \Rightarrow s(sX_1(s)) = -2(X_1(s)) - (sX_1(s)) + U$$

$$sX_3(s) = a\, X_3(s)$$

$$\Rightarrow U(s) = (s^2 + s + 2)\, X_1(s)$$

$$\Rightarrow \frac{X_1(s)}{U(s)} = \frac{1}{s^2+s+2}$$

Since $Y(s) = X_1(s)$, we get

$$\frac{Y(s)}{U(s)} = \frac{X_1(s)}{U(s)} = \frac{1}{s^2+s+2}.$$

[Note: $x_3$ plays no role; since neither the input $u$ affects $x_3$ (even indirectly through another state) nor $x_3$ affect the output $y$ (even indirectly). That is why the transfer function from $U(s)$ to $Y(s)$ is second order (only the dynamics of $x_1$ & $x_2$ have an impact in the T.F.) even though the state dimension is three.]

Problem 7 states are position and velocity.

call $p(t) =$ position

$v(t) =$ velocity.

$\dot{p} = v \longleftarrow$ (by definition of $p$ & $v$)

$\dot{v} = -av + b\theta \longleftarrow$ given

$$\Rightarrow \begin{bmatrix} \dot{p} \\ \dot{v} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix}}_{\mathbb{A}} \begin{bmatrix} p \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} \theta$$

$\mathbb{X}$ = state vector = $\begin{bmatrix} p \\ v \end{bmatrix}$. $\mathbb{U}$ = input vector = $[\theta]$

$\mathbb{Y}$ = measured output = $v$ (given)

$$= \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{\mathbb{C}} \begin{bmatrix} p \\ v \end{bmatrix}$$

reconstructibility :

$$\mathbb{P}_o = \begin{bmatrix} \mathbb{C} \\ \mathbb{C}\mathbb{A} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix} \Rightarrow \det(\mathbb{P}_o) = 0 \Rightarrow \text{rank}(\mathbb{P}_o) < 2$$

$\Rightarrow (\mathbb{A}; \mathbb{C})$ is not completely reconstructible.

Note 1 : The elements of the state vector $\mathbb{X}$ are usually written as $x_1, x_2 \ldots x_n$, but there is nothing sacred about that notation. In this example, $p$ and $v$ are the states.

Note 2: It makes sense that the state vector $\mathbb{X}(t)$ cannot be uniquely determined since ~~the~~ the velocity is measured, since the position cannot be determined from measurement of velocity. If the position were measured, the system would have been found to be reconstructible. Think about it.

Problem 8

$$\mathbb{P}_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} \qquad C = [1 \;\; 0 \;\; 0]$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & a \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 0 \end{bmatrix}$$

$\det(\mathbb{P}_o) = 0$ (since the last column is $\underline{0}$, you can see the det is 0 without computing it)

$\Rightarrow$ rank $(\mathbb{P}_o) < 3$

$\Rightarrow$ $(A, C)$ is not completely reconstructible.

Note: Again, the answer makes intuitive sense since the state $x_3$ doesnot even indirectly affect the output $y$. There is no hope of determing $x_3$ from $y$ and $u$.

Problem 9.1 For a system $\dot{x} = Ax + Bu$, $y = Cx$,

the poles of the T.F. from $u$ to $y$ are the eigenvalues of $A$.

$\text{eig}(A) =$ roots of $\det(\lambda I - A) = 0$

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -0.02 & -0.2 \end{bmatrix}\right)$$

$$= \det\begin{pmatrix} \lambda & -1 \\ 0.02 & \lambda + 0.2 \end{pmatrix}$$

$$= \lambda(\lambda + 0.2) + 0.02$$

$$= \lambda^2 + 0.2\lambda + 0.02$$

$\therefore$ eigenvalues : $\lambda^2 + 0.2\lambda + 0.02 = 0$

$$\Rightarrow \lambda = \frac{-0.2 \pm \sqrt{(0.2)^2 - 4 \times 0.02}}{2}$$

$$= -0.1 \pm 0.1j$$

9.2 Closed loop eigenvalues can be placed anywhere we want if ~~the~~ $(A, B)$ is completely controllable.

Check: $P_c = [B \quad AB] = \begin{bmatrix} 0 & 1 \\ 1 & -0.2 \end{bmatrix}$ : rank 2.
$\Rightarrow$ controllable.

So, yes, the closed loop eigenvalues can be placed at $-1 \pm 0.1j$ (or anywhere we want) by state feedback.

To design the state feedback controller ($u = -\mathbb{K}x$), that is, to design the gain $\mathbb{K}$, so that the eigenvalues of $\mathbb{A} - \mathbb{B}\mathbb{K}$ are at $-1 \pm 0.1j$, we use Ackerman's formula.

step 1 : define $q(s) = (s - \lambda_1^{(d)})(s - \lambda_2^{(d)})$

$$= [s - (-1 + 0.1j)][s - (-1 - 0.1j)]$$

$$= s^2 + 2s + 1.01$$

$$\Rightarrow q(\mathbb{A}) = \mathbb{A}^2 + 2\mathbb{A} + 1.01 \cdot I_{2\times 2}$$

$$= \begin{bmatrix} -0.02 & -0.2 \\ 0.004 & 0.02 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ -0.02 & -0.2 \end{bmatrix} + 1.01\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.99 & 1.8 \\ -0.036 & 0.63 \end{bmatrix}$$

step 2 : determine $\mathbb{P}_c^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -0.2 \end{bmatrix}^{-1}$

$$= \frac{1}{-1}\begin{bmatrix} -0.2 & -1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & 1 \\ 1 & 0 \end{bmatrix}$$

step 3 : Apply Ackerman's formula: $\mathbb{K} = [0 \;\; 1]\, \mathbb{P}_c^{-1}\, q(\mathbb{A})$

$$\Rightarrow \mathbb{K} = [0 \;\; 1]\begin{bmatrix} 0.2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0.99 & 1.8 \\ -0.036 & 0.63 \end{bmatrix}$$

$$\Rightarrow K = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.99 & 1.8 \\ -0.036 & 0.63 \end{bmatrix}$$

$$= \begin{bmatrix} 0.99 & 1.8 \end{bmatrix}$$

<u>check:</u> Are the eigenvalues of $A - BK$ really at $-1 \pm 0.1j$ ?

$$A - BK = \begin{bmatrix} 0 & 1 \\ -0.02 & -0.2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0.99 & 1.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -0.02 & -0.2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0.99 & 1.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -1.01 & -2 \end{bmatrix}$$

eigenvalues of $A - BK$ can be computed now to show that they are indeed $-1 \pm 0.1j$

## Problem 10

$$m\ddot{h} = -F(i,h) + mg + w$$

$$F(i,h) = \frac{\mu_0 N^2 A}{4} \cdot \frac{i^2}{h^2}$$

**10·1** to determine $i_0$ so that ~~...~~

$mg = F(i_0, h_0)$, set

$$\frac{\mu_0 N^2 A}{4} \frac{i_0^2}{h_0^2} = mg$$

$$\Rightarrow \boxed{i_0 = h_0 \sqrt{mg \Big/ \frac{\mu_0 N^2 A}{4}}}$$ (ignore the negative root)

$(= 26{\cdot}061 \text{ Amp})$

When $w(t) \equiv 0$ ($\overset{*}{0}$ for all $t$), $i(t) = i_0$ $t \geq \bar{t}$,

and $h(\bar{t}) = 0$, then

$$m\,\ddot{h}(\bar{t}) = -F\big(i(\bar{t}), h(\bar{t})\big) + mg + 0$$

$$= -F(i_0, h_0) + mg$$

$$= 0.$$ (since $i_0$ by definition

satisfies $mg = F(i_0, h_0)$)

Since $\ddot{h}(\bar{t}) = 0$, both ~~■~~ $h(t)$ and $\dot{h}(t)$ will not change

for $t \geq \bar{t}$ $\Rightarrow$ $h(t) = h_0$ for all $t \geq \bar{t}$.

10.2 $\quad \tilde{i} = i - i_0$, $\quad x_1 = h, \quad x_2 = \dot{h}$ : states

$\tilde{i}, \omega$ : external inputs.

to find state space description, find

$$\dot{x}_1 = f_1(x_1, x_2, \tilde{i}, \omega) \ ?$$
$$\dot{x}_2 = f_2(x_1, x_2, \tilde{i}, \omega) \ ?$$

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$\dot{x}_1 = \dot{h} = x_2$ by definition. $\longleftarrow f_1$

$$\dot{x}_2 = \ddot{h} = \frac{1}{m}\left(-\frac{\mu_0 N^2 A}{4}\cdot\frac{i^2}{h^2} + mg + w\right)$$

$$= \frac{1}{m}\left(-\frac{\mu_0 N^2 A}{4}\frac{(i_0^2 + \tilde{i})^2}{x_1^2} + mg + w\right). \longleftarrow f_2$$

this is not a linear model, since it cannot be expressed as $\dot{\mathbb{X}} = \mathbb{A}\mathbb{X} + \mathbb{B}\mathbb{U}$, where

$\mathbb{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbb{U} = \begin{bmatrix} \tilde{i} \\ w \end{bmatrix}$. The reason is that

$f_2$ is a non-linear function of $x_1, \tilde{i}$.

10.3 $\quad$ To find equilibria of $\dot{\mathbb{X}} = f(\mathbb{X}, \mathbb{U})$,

set $f(\mathbb{X}^*, 0) = 0$ and solve for $\mathbb{X}^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$

$$\Rightarrow \begin{cases} f_1(x_1^*, x_2^*, 0, 0) = 0 \Rightarrow \boxed{x_2^* = 0} \\ f_2(x_1^*, x_2^*, 0, 0) = 0 \Rightarrow -\dfrac{\mu_0 N^2 A}{4}\dfrac{i_0^2}{x_1^{*2}} + mg = 0 \end{cases}$$

$\Rightarrow x_1^* = h_0$ (use the formula for $i_0$ to figure this out. I am also ignoring $x_1^* = -h_0$, which is another solution).

$\Rightarrow$ An equilibrium point of the system is

$$\mathbb{X}^* = \begin{bmatrix} h_0 \\ 0 \end{bmatrix}.$$

10.4:

$$\tilde{x}_1 = x_1 - x_1^* = x_1 - h_0$$

$$\tilde{x}_2 = x_2 - x_2^* = x_2$$

To linearize, we ~~express equa~~ find equations for the ~~deviations~~ $\dot{\tilde{x}}_1$ and $\dot{\tilde{x}}_2$ and simplify them by assuming that the deviations $\tilde{x}_1$ and $\tilde{x}_2$ are small.

$$\dot{\tilde{x}}_1 = \frac{d}{dt}(x_1 - x_1^*) = \dot{x}_1 = x_2 = \tilde{x}_2 \Rightarrow \boxed{\dot{\tilde{x}}_1 = \tilde{x}_2} \text{ — ⓐ}$$

$$\dot{\tilde{x}}_2 = \frac{1}{m}\left(-\frac{\mu_0 N^2 A}{4}\frac{(i_0+\tilde{i})^2}{(h_0+\tilde{x}_1)^2} + mg + w\right)$$

$$= \frac{1}{m}\left[-\frac{\mu_0 N^2 A}{4}\frac{i_0^2}{h_0^2}\frac{(1+\tilde{i}/i_0)^2}{(1+\tilde{x}_1/h_0)^2} + mg + w\right]$$

$$= \frac{1}{m}\left[-\frac{\mu_0 N^2 A}{4}\frac{i_0^2}{h_0^2}\left(1+\frac{\tilde{i}}{i_0}\right)^2\left(1+\frac{\tilde{x}_1}{h_0}\right)^{-2} + mg + w\right]$$

$$= \frac{1}{m}\left[-mg\left(1 + 2\frac{\tilde{i}}{i_0} + \frac{\tilde{i}^2}{i_0^2}\right)\left(1 - 2\frac{\tilde{x}_1}{h_0} + O\left(\frac{\tilde{x}_1^2}{h_0^2}\right)\right) + mg + w\right]$$

$\uparrow$ since $F_0 = mg$

$$\Rightarrow \dot{\tilde{x}}_2 = \frac{1}{m}\Big[-mg\Big(1 - 2\frac{\tilde{x}_1}{h_0} + 2\frac{\tilde{i}}{i_0} + \text{terms involving } \left(\frac{\tilde{i}}{i_0}\right)^2,$$

$$\left(\frac{\tilde{x}_1}{h_0}\right)^2, \left(\frac{\tilde{i}}{i_0}, \frac{\tilde{x}_1}{h_0}\right), \text{ and even higher order}\Big)$$

$$+ mg + w\Big]$$

$$= \frac{1}{m}\Big[-\cancel{mg} + 2mg\left(\frac{\tilde{x}_1}{h_0} - \frac{\tilde{i}}{i_0}\right) + O\left(\left(\frac{\tilde{x}_1}{h_0}\right)^2, \left(\frac{\tilde{i}}{i_0}\right)^2, \ldots\right)$$

$$+ \cancel{mg} + w\Big]$$

$$\approx \frac{1}{\cancel{m}}\left[2\cancel{m}g\left(\frac{\tilde{x}_1}{h_0} - \frac{\tilde{i}}{i_0}\right)\right] + \frac{w}{m} \quad \left(\text{assuming } \left|\frac{\tilde{x}_1}{h_0}\right| \ll 1, \left|\frac{\tilde{i}}{i_0}\right| \ll 1\right)$$

$\therefore$ linear approximation of $\dot{\tilde{x}}_2$ is :

$$\boxed{\dot{\tilde{x}}_2 = \frac{2g}{h_0}\tilde{x}_1 - \frac{2g}{i_0}\tilde{i}} \quad \text{— (b)}$$

(a) and (b) can now be written in the form:

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{2g}{h_0} & 0 \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{2g}{i_0} \end{bmatrix}\tilde{i} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} w$$

$$= \begin{bmatrix} 0 & 1 \\ \frac{2g}{h_0} & 0 \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\frac{2g}{i_0} & \frac{1}{m} \end{bmatrix}\begin{bmatrix} \tilde{i} \\ w \end{bmatrix}$$

Since we did not decide what is being measured, there is no $y$. (so, <u>no</u> $y = Cx$)

10.5 In matlab, a LTI system $\dot{x} = Ax + Bu$, $y = Cx + Du$ can be defined using ss (A, B, C, D).

In our case, since output $y$ is not specified, we arbitrarily choose $y = x$ (full state as measured output)

so in matlab you use $C = \text{eye}(2)$, $D = \text{zeros}(2)$.

Since you are asked to simulate with $i(t) = i_0$ for all $t$, it means $\tilde{i}(t) = 0$ for all time, (and also $w(t) \equiv 0$). So $u(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for all $t$.

The system is unstable, (eigenvalues of A are $\pm 49.4975$) so the states will blow up.

See the code provided in the website.

Note: it might appear that states stay close to 0 until the very end, then at the end they shoot up to very large numbers. This impression is created because of the scale of the y-axis in the plots.
If you plot in log scale you'll see that the states are continuously increasing, at all times.

(see the code)

Problem 11 $\qquad \dfrac{Y}{U} = \dfrac{1}{s^2 + 2s}$

$$\Rightarrow \ddot{y} + 2\dot{y} = u$$

states $\quad x_1 = y$

$\qquad\qquad x_2 = \dot{y}$

$$\Rightarrow \quad \dot{x}_1 = x_2$$

$$\dot{x}_2 = \ddot{y} = u - 2\dot{y} = u - 2x_2$$

$$\Rightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}}_{\mathbb{A}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = x_1 = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbb{C}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

observer: $\qquad \hat{\mathbb{X}}(t)$: estimate of the state $\mathbb{X}(t)$

$$\dot{\hat{\mathbb{X}}} = \mathbb{A}\hat{\mathbb{X}} + \mathbb{B}u + \mathbb{L}(y - \mathbb{C}\hat{\mathbb{X}})$$

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} (y - \hat{x}_1), \quad \left( \begin{matrix} \hat{\mathbb{X}}(0) \\ \text{arbitrary} \end{matrix} \right)$$

$\mathbb{L}$, (that is, $l_1$ and $l_2$) need to be designed so that the eigenvalues of $A - LC$ are at $-3 \pm 4j$

Use Ackerman's formula for observer design:

$$\mathbb{L} = q(A) \cdot \mathbb{P}_o^{-1} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

steps : find $q(A)$, $\mathbb{P}_o^{-1}$ & then compute $L$

$$q(s) = \left(s - \lambda_1^{(d)}\right)\left(s - \lambda_2^{(d)}\right)$$

desired eigenvalues of $A - LC$

$$\Rightarrow \lambda_1^d = -3 + 4j \quad \text{(specified)}$$
$$\lambda_2^d = -3 - 4j$$

$$= \left(s + 3 + 4j\right)\left(s + 3 - 4j\right) = s^2 + 6s + 25$$

$$\Rightarrow q(A) = A^2 + 6.A + 25.I = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}^2 + 6\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 6 \\ 0 & -12 \end{bmatrix} + \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 4 \\ 0 & 17 \end{bmatrix} \checkmark$$

$$\mathbb{P}_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow P_o^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow L = \begin{bmatrix} 25 & 4 \\ 0 & 17 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 25 & 4 \\ 0 & 17 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 17 \end{bmatrix}$$

Now check if the eigenvalues of $A - LC$ are indeed $-3 \pm 4j$

$$A - LC = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 4 \\ 17 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 17 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ -17 & -2 \end{bmatrix}$$

eigenvalues of $\begin{bmatrix} -4 & 1 \\ -17 & -2 \end{bmatrix}$ are indeed $-3 \pm 4j$. done!