Problem 1

$$\mathcal{L}(e^{Pt}) = \int_0^\infty e^{Pt} e^{-st}\,dt = \int_0^\infty e^{(P-s)t}\,dt$$

$$= \frac{1}{P-S} e^{(P-s)t}\Big|_0^\infty = \frac{1}{P-S}\left(\lim_{t\to\infty} e^{(P-s)t} - 1\right)$$

$$= \frac{1}{S-P}\left(1 - \lim_{t\to\infty} e^{(P-s)t}\right)$$

$\lim_{t\to\infty} e^{(P-s)t}$ exist or not ?

Here, $s = \sigma + j\omega$, and let $p = \alpha + j\beta$

$$\lim_{t\to\infty} e^{[\alpha-\sigma+j(\beta-\omega)]t} = \lim_{t\to\infty} e^{(\alpha-\sigma)t} \cdot e^{j(\beta-\omega)t}$$

case 1: $\alpha - \sigma < 0$, which means

$\lim_{t\to\infty} e^{(\alpha-\sigma)t} = 0$ and $e^{j(\beta-\omega)t}$ is bounded

so, $\lim_{t\to\infty} e^{(\alpha-\sigma)t} \cdot e^{j(\beta-\omega)t} = 0$

case 2: $\alpha - \sigma = 0$

$\lim_{t\to\infty} e^{(\alpha-\sigma)t} = 1$ and $e^{j(\beta-\omega)t} = \cos(\beta-\omega)t + j\sin(\beta-\omega)t$, which does not have a limit.

so, $\lim_{t\to\infty} e^{(\alpha-\sigma)t} \cdot e^{j(\beta-\omega)t}$ does not exist.

case 3: $\alpha - \sigma > 0$

$\lim_{t\to\infty} e^{(\alpha-\sigma)t}$ is ∞ and $e^{j(\beta-\omega)t}$ oscillates

$\Rightarrow \lim_{t\to\infty} e^{(\alpha-\sigma)t} \cdot e^{j(\beta-\omega)t}$ does not exist.

All in all, only when $\sigma > \alpha$, the Laplace transform is well defined.

$$\mathcal{L}(e^{pt}) = \frac{1}{s-p} \quad , \quad Re(s) > Re(p)$$

oblem 2.

1) $\mathcal{L}(\delta(t)) = \int_{0^-}^{\infty} \delta(t)\, e^{-st}\, dt = e^{-s.0} = e^0 = 1$

2) $\mathcal{L}(1(t)) = \int_0^{\infty} 1(t)\, e^{-st}\, dt = -\frac{1}{s} e^{-st}\Big|_0^{\infty}$

$= -\frac{1}{s}(0-1) = \frac{1}{s} \quad , \quad Re(s) > 0$

3) $\mathcal{L}(\sin(w_0 t)) = \int_0^{\infty} \sin(w_0 t) \cdot e^{-st}\, dt = \frac{1}{2j}\int_0^{\infty} (e^{jw_0 t} - e^{-jw_0 t})\, e^{-st} dt$

$= \frac{1}{2j}\int_0^{\infty} e^{jw_0 t - st} dt - \frac{1}{2j}\int_0^{\infty} e^{-jw_0 t - st} dt$

$= \frac{1}{2j} \cdot \frac{1}{s - jw_0} - \frac{1}{2j} \cdot \frac{1}{s+jw_0} = \frac{1}{2j} \cdot \frac{s + jw_0 - s + jw_0}{(s-jw_0)(s+jw_0)}$

$= \frac{w_0}{s^2 + w_0^2}$

4) $\mathcal{L}(\cos(w_0 t)) = \int_0^{\infty} \cos(w_0 t)\, e^{-st}\, dt = \frac{1}{2}\int_0^{\infty} (e^{jw_0 t} + e^{-jw_0 t})\, e^{-st}\, dt$

$= \frac{1}{2}\int_0^{\infty} e^{jw_0 t - st} dt + \frac{1}{2}\int_0^{\infty} e^{-jw_0 t - st} dt$

$= \frac{1}{2} \cdot \frac{1}{s - jw_0} + \frac{1}{2} \frac{1}{s + jw_0} = \frac{s}{s^2 + w_0^2}$

5) $x(t) \xrightarrow{\mathcal{L}} X(s) \quad , \quad y(t) \xrightarrow{\mathcal{L}} Y(s)$

$\mathcal{L}(\alpha x(t) + \beta y(t)) = \int_0^{\infty} (\alpha x(t) + \beta y(t))\, e^{-st}\, dt$

$= \alpha \int_0^{\infty} x(t)\, e^{-st}\, dt + \beta \int_0^{\infty} y(t)\, e^{-st}\, dt$

$= \alpha X(s) + \beta Y(s)$

6) $y(t) \xrightarrow{\mathcal{L}} Y(s)$, $z(t) \triangleq \dot{y}(t)$

$$\mathcal{L}(z(t)) = \int_0^\infty z(t)\, e^{-st}\, dt = \int_0^\infty \dot{y}(t)\, e^{-st}\, dt = \int_0^\infty e^{-st}\, dy(t)$$

$$= e^{-st} y(t) \Big|_0^\infty - \int_0^\infty y(t)\, e^{-st} (-s)\, dt$$

$$= \underbrace{0}_{} - y(0) + s \int_0^\infty y(t)\, e^{-st}\, dt = sY(s) - y(0)$$

→ see notes in website for why $\lim_{t\to\infty} e^{-st} y(t) = 0$

Problem 3

1) $y(t) \xrightarrow{\mathcal{L}} Y(s)$, $z(t) \triangleq \int_0^t y(\tau)\, d\tau$

$$\Rightarrow \dot{z}(t) = y(t)$$

$$\Rightarrow s Z(s) - z(0) = Y(s)$$

$$z(0) = \int_0^0 y(\tau)\, d\tau = 0$$

$$\Rightarrow Z(s) = \frac{1}{s} \cdot Y(s)$$

Problem 3 (contd.)

2) $t(t) = \int_0^t 1(\tau)\,d\tau$, $\mathcal{L}(1(t)) = \frac{1}{s}$

$$\Rightarrow \mathcal{L}(t(t)) = \mathcal{L}\left(\int_0^t 1(\tau)\,d\tau\right) = \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2}$$

3) $\frac{d}{dt}(t\cdot e^{pt}) = e^{pt} + pt\,e^{pt}$ — (*)

$$\mathcal{L}\left[\frac{d}{dt}(t\cdot e^{pt})\right] = s\,\mathcal{L}(t\cdot e^{pt}) - 0$$

$$(*) \Rightarrow \mathcal{L}\left[\frac{d}{dt}(t\cdot e^{pt})\right] = \mathcal{L}(e^{pt} + pt\,e^{pt}) = \frac{1}{s-p} + p\,\mathcal{L}(t\cdot e^{pt})$$

$$\Rightarrow s\,\mathcal{L}(t\cdot e^{pt}) = \frac{1}{s-p} + p\,\mathcal{L}(t\cdot e^{pt})$$

$$\Rightarrow \mathcal{L}(t\cdot e^{pt}) = \frac{1}{(s-p)^2}$$