Solution to HW9
EML 4312

Problem 1

$$2\ddot{y}_1 + (\dot{y}_1 - \dot{y}_2) + 10(y_1 - y_2) = 4u$$

$$\Rightarrow \ddot{y}_1 + \tfrac{1}{2}(\dot{y}_1 - \dot{y}_2) + 5(y_1 - y_2) = 2u$$

$$\Rightarrow \left(s^2 + \tfrac{1}{2}s + 5\right)Y_1(s) + \left(-\tfrac{1}{2}s - 5\right)Y_2(s) = 2\,U(s)$$

second equation : $\ddot{y}_2 + 2\dot{y}_1 + 3\dot{y}_1 + 6y_2 - 4y_1 = f - u$

$$\Rightarrow (s^2 + 6)\,Y_2(s) + (5s - 4)\,Y_1(s) = F(s) - U(s)$$

$$\Rightarrow \underbrace{\begin{bmatrix} (s^2 + \frac{1}{2}s + 5) & -\left(\frac{s}{2} + 5\right) \\ 5s - 4 & s^2 + 6 \end{bmatrix}}_{M(s)} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} 2\,U(s) \\ F(s) - U(s) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} U \\ F \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = M^{-1} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} U(s) \\ F(s) \end{bmatrix}$$

$$M^{-1} = \frac{1}{\det(M)}\,\mathrm{adj}(M)$$

$$\det(M) = \left(s^2 + \tfrac{s}{2} + 5\right)(s^2 + 6) + \left(\tfrac{s}{2} + 5\right)(5s - 4)$$

$$= s^4 + \tfrac{1}{2}s^3 + 5s^2 + 6s^2 + 3s + 30 + 2.5s^2 - 2s + 25s - 20$$

$$= s^4 + \tfrac{1}{2}s^3 + 13.5s^2 + \boxed{26}\,s + 10$$

call it $D(s)$.

$$\mathrm{adj}(M) = \begin{bmatrix} s^2+6 & -(5s-4) \\ \frac{s}{2}+5 & s^2+\frac{1}{2}s+5 \end{bmatrix}^T$$

$$\Rightarrow \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \frac{1}{D(s)} \begin{bmatrix} s^2+6 & \frac{s}{2}+5 \\ -5s+4 & s^2+\frac{1}{2}s+5 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} U \\ F \end{bmatrix}$$

$$= \frac{1}{D(s)} \begin{bmatrix} 2s^2+12-\frac{s}{2}-5 & \frac{s}{2}+5 \\ -10s+8-s^2-\frac{1}{2}s-5 & s^2+\frac{1}{2}s+5 \end{bmatrix} \begin{bmatrix} U \\ F \end{bmatrix}$$

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{2s^2-\frac{1}{2}s+7}{D(s)} & \frac{\frac{1}{2}s+5}{D(s)} \\ -\frac{s^2-10.5s+3}{D(s)} & \frac{s^2+\frac{1}{2}s+5}{D(s)} \end{bmatrix} \begin{bmatrix} U(s) \\ F(s) \end{bmatrix}$$

where $D(s) = s^4 + \frac{1}{2}s^3 + 13.5s^2 + 24s + 10$ Ans.

Problem 2 First find the equilibrium point by setting $f(x^*, 0) = 0$

$$\Rightarrow x_2^* = 0, \text{ and}$$

$$-\frac{g}{m}x_2^* - \frac{g}{l}\sin x_1^* + 0 = 0$$

$$\Rightarrow \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 0 \end{bmatrix}, \begin{bmatrix} -\pi \\ 0 \end{bmatrix}, \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}, \ldots$$

We are asked to linearize around $\begin{vmatrix} 0 \\ 0 \end{vmatrix}$

$\therefore$ Define $\quad \tilde{x}_1 \triangleq x_1 - x_1^* = x_1 - 0 = x_1$

$$\tilde{x}_2 = x_2 - x_2^* = x_2 - 0 = x_2$$

assume $\tilde{x}_1, \tilde{x}_2$ are small, that is x_1, x_2 are small.

$\dot{x}_1 = x_2 \Rightarrow \dot{\tilde{x}}_1 = \tilde{x}_2$ (linear, ✓)

$$\dot{x}_2 = -\frac{g}{m} x_2 - \frac{g}{l} \sin x_1 + u$$

$$\Rightarrow \dot{\tilde{x}}_2 = -\frac{g}{m} \tilde{x}_2 - \frac{g}{l} \sin \tilde{x}_1 + u$$

$$\approx -\frac{g}{m} \tilde{x}_2 - \frac{g}{l} \tilde{x}_1 + u,$$ since $\sin \theta \approx \theta$ for small θ.

this is linear. So,

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{g}{m} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Ans.

To check stability, compute eigenvalues of A:

$$\det(\lambda I - A) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} \lambda & -1 \\ \frac{g}{l} & \lambda + \frac{g}{m} \end{bmatrix}\right) = 0 \Rightarrow \lambda^2 + \frac{g}{m}\lambda + \frac{g}{l} = 0$$

since $\frac{g}{m}, \frac{g}{l}$ are positive, roots of this equation are in the left half complex plane. So the system is stable.

Problem 3

$$\frac{Y}{U} = \frac{1}{s^3 + 3s^2 + 5s + 56}$$

$$\Rightarrow (s^3 + 3s^2 + 5s + 56)\,Y(s) = U(s)$$

$$\Rightarrow \dddot{y} + 3\ddot{y} + 5\dot{y} + 56y = u(t)$$

Define $x_1 = y$

$x_2 = \dot{x}_1$

$x_3 = \dot{x}_2$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = \ddot{x}_2 = \dddot{x}_1 = \dddot{y} = u - 3\ddot{y} - 5\dot{y} - 56y$$

$$= u - 3\ddot{x}_1 - 5\dot{x}_1 - 56x_1$$

$$= u - 3\dot{x}_2 - 5x_2 - 56x_1$$

$$= u - 3x_3 - 5x_2 - 56x_1$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -56 & -5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Ans

Problem 4 rate of momentum = net external force

$$\Rightarrow \frac{d}{dt}(m\dot{y}) = k\frac{dm}{dt} - mg$$

$$\Rightarrow \dot{m}\dot{y} + m\ddot{y} = k\dot{m} - mg$$

(1) Define states $x_1 = y$

$x_2 = \dot{x}_1 = \dot{y}$

$x_3 = m$

and control $u = \dot{m} = \dot{x}_3$

(2) Find equations for $\dot{x}_1, \dot{x}_2, \dot{x}_3$:

$\dot{x}_1 = x_2$

$$\dot{x}_2 = \ddot{y} = \frac{1}{m}(k\dot{m} - mg - \dot{m}\dot{y})$$

$$= \frac{1}{m}(\dot{m}(k-\dot{y})) - g$$

$$= \frac{1}{x_3}(u(k-x_2)) - g$$

$\dot{x}_3 = \dot{m} = u$

$$\Rightarrow \left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{k-x_2}{x_3}\cdot u - g \\ \dot{x}_3 &= u \end{aligned}\right]$$

clearly not a linear equation, since $\dot{\mathbb{X}}$ cannot be written as $\mathbb{A}\mathbb{X} + \mathbb{B}.\mathbb{U}$.

Problem 5 considering the pendulum, we get

$$ml^2\left(\ddot{\theta} + \frac{\ddot{x}\cos\theta}{l}\right) = \left(mg\cos(90-\theta)\right)l$$

$$\Rightarrow l\ddot{\theta} + \cos\theta\,\ddot{x} - g\sin\theta = 0 \quad \text{—— ①}$$

Considering the cart : $M\ddot{x} = -mg\sin\theta + u$

$$\Rightarrow \ddot{x} = -\frac{m}{M}g\sin\theta + \frac{1}{M}u$$

① Define states: $x_1 = x$

$$x_2 = \dot{x}_1 \; (= \dot{x})$$

$$x_3 = \theta$$

$$x_4 = \dot{x}_3 \; (= \dot{\theta})$$

② Find $\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \ddot{x} = -\frac{m}{M}g\sin x_3 + \frac{1}{M}u$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \ddot{\theta} = \frac{1}{l}\left(g\sin\theta - \ddot{x}\cos\theta\right)$$

$$= \frac{1}{l}\left(g\sin x_3 - \dot{x}_2\cos x_3\right)$$

$\dot{x}_2$ has to be eliminated, we cannot have derivatives on the RHS

$$= \frac{1}{l}\left(g\sin x_3 - \left(-\frac{m}{M}g\sin x_3 + \frac{1}{M}u\right)\cos x_3\right)$$

$$= \frac{g}{l}\sin x_3 + \frac{mg}{Ml}\sin x_3\cos x_3 - \frac{\cos x_3}{Ml}u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{m}{M} g \sin x_3 + \frac{1}{M} u \\ x_4 \\ \frac{g}{l} \sin x_3 + \frac{mg}{2Ml} \sin 2x_3 - \frac{1}{Ml} \cos x_3 \cdot u \end{bmatrix} = f(\mathbb{X}, u)$$

Clearly, not a linear model.

5.2 When $x_1(t) = x_0$, $x_2(t) = 0$, $x_3(t) = 0$, $x_4(t) = 0$,

$$f(\mathbb{X}^*, 0) = \begin{pmatrix} 0 \\ -0+0 \\ 0 \\ 0+0-\frac{1}{Ml} \cdot 1 \cdot 0 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \mathbb{X}^* = [x_0, 0, 0, 0]^T$ is an equilibrium point.

physically, this equilibrium point corresponds to the pendulum being vertically upright with 0 angular velocity, the cart being at position x_0 with 0 linear velocity, and the external ~~fo~~ torque u being 0.

5.3 Linearization around $\mathbb{X}^* = (0, 0, 0, 0)^T$.

define
$$\tilde{x}_1 = x_1 - x_1^* = x_1 - 0 = x_1$$
$$\tilde{x}_2 = x_2 - x_2^* = x_2$$
$$\tilde{x}_3 = x_3 - x_3^* = x_3$$
$$\tilde{x}_4 = x_4 - x_4^* = x_4$$

assume $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4$ small

$\sin x_3 = \sin(\tilde{x}_3) \simeq \tilde{x}_3 \quad \because \tilde{x}_3$ is small

$\cos x_3 = \cos(\tilde{x}_3) \approx 1$

$$\Rightarrow \quad \dot{\tilde{x}}_1 = \tilde{x}_2$$

$$\left.\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 \\ \dot{\tilde{x}}_2 &\approx -\frac{mg}{M}\tilde{x}_3 + \frac{1}{M}u \\ \dot{\tilde{x}}_3 &= \tilde{x}_4 \\ \dot{\tilde{x}}_4 &\approx \frac{g}{l}\tilde{x}_3 + \frac{mg}{Ml}\tilde{x}_3 - \frac{1}{Ml}u \end{aligned}\right\} \text{linear approximation}$$

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \\ \dot{\tilde{x}}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{l}\left(1+\frac{m}{M}\right) & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{Ml} \end{bmatrix} u$$

5.4 For $l=1$, $m=0.1$, $M=1$,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -.98 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 10.78 & 0 \end{bmatrix}$$

eig(A) in matlab yields: 0, 0, 3.2833, −3.2833

Only one of the four eigenvalues are in the LHP.

$\Rightarrow$ unstable

Problem 6: Was done in class in the beginning of the course for the Maglev example.

In addition, it is in HW 10

so no solution is provided