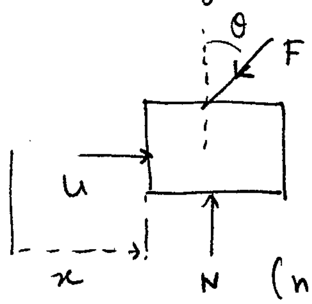
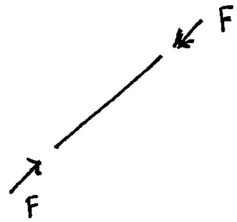
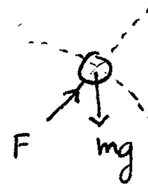
Free body diagrams:



(normal reaction)





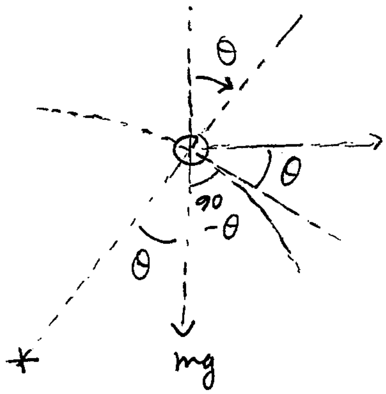
Consider the pendulum first:

(angular acceleration)(moment of inertia) = net moment

$$\left(\ddot{\theta} + \frac{\ddot{x}}{l}\cos\theta\right).ml^2 = (mg\sin\theta)\, l$$

↑

this is the contribution to the angular acceleration of the pendulum about the hinge point due to its linear acceleration $\ddot{x}$ imparted by the cart.



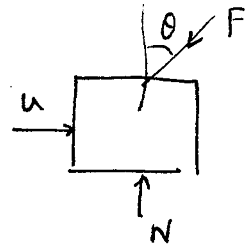
$\phi = 90 - \theta$

$\Rightarrow mg\cos\phi = mg\sin\theta$

$$\Rightarrow \ddot{\theta} + \frac{\ddot{x}\cos\theta}{l} = \frac{g\sin\theta}{l}$$

$$\Rightarrow \ddot{\theta} + \ddot{x}\frac{\cos\theta}{l} - \frac{g\sin\theta}{l} = 0 \qquad \text{— (1)}$$

Now consider the cart :

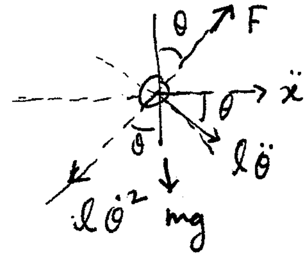


$$M\ddot{x} = u - F\sin\theta \qquad \text{—— ②}$$

($F\cos\theta$ balances N)

So we need F. To find F,

Consider the pendulum again, but this time look at linear acceleration in the horizontal direction :



Net acceleration in the horizontal direction $= \underbrace{\ddot{x} - l\dot{\theta}^2\cos(90^\circ - \theta)}_{\uparrow \text{ centripetal acceleration}} + l\ddot{\theta}\cos\theta$

$$\Rightarrow m\left(\ddot{x} - l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta\right) = F\sin\theta \qquad \text{—— ③}$$

using ③ in ②, we get

$$M\ddot{x} = u - m\left(\ddot{x} - l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta\right)$$

$$\Rightarrow (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = u \qquad \text{—— ④}$$

So the two differential equations that describe the dynamics of the system are

$$\left.\begin{aligned} &\ddot{\theta} + \frac{1}{l}\cos\theta\,\ddot{x} - \frac{g}{l}\sin\theta = 0 \\ &(M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = u \end{aligned}\right]$$

To express this system in state space form,

① Define states: $x_1 = x$

$$x_2 = \dot{x}_1 = \dot{x}$$

$$x_3 = \theta$$

$$x_4 = \dot{\theta}$$

② Determine $\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4$:

$\dot{x}_1 = x_2$, $\dot{x}_3 = x_4$ by definition

$$\dot{x}_2 = \ddot{x} = ? \quad \left(\text{from } \ddot{\theta} + \frac{1}{l}\cos\theta\,\ddot{x} - \frac{g}{l}\sin\theta = 0\right),$$

$$= \left(\frac{g}{l}\sin\theta - \ddot{\theta}\right)\frac{l}{\cos\theta}$$

$$= \left(\frac{g}{l}\sin x_3 - \dot{x}_4\right)\frac{l}{\cos x_3}.$$

this is not what we wanted, since right hand side contains a derivative of a state.

Instead, lets use the second equation to get

$$\ddot{x} = \frac{1}{M+m}\left(u + ml\,x_4^2\sin x_3 - ml\cos x_3\,\dot{x}_4\right)$$

this is useless, too. But, $\dot{x}_4 = \ddot{\theta} = \frac{g}{l}\sin\theta - \frac{1}{l}\cos\theta.\,\ddot{x}$

$$\Rightarrow \ddot{x} = \frac{1}{M+m}\left(u + ml\,x_4^2\sin x_3 - ml\cos x_3\,\frac{1}{l}\left(g\sin\theta - \cos\theta\,\ddot{x}\right)\right)$$

$$\Rightarrow \left((M+m) - m\cos^2 x_3\right)\ddot{x} = u + ml\,x_4^2\sin x_3 - m\cos x_3\cdot g\cdot\sin x_3$$

$$\Rightarrow \dot{x}_2 = \ddot{x} = \frac{1}{(M+m) - m\cos^2 x_3}\left[ml\,x_4^2\sin x_3 - mg\cos x_3\sin x_3 + u\right]$$

and, $\dot{x}_4 = \ddot{\theta} = \frac{1}{l}\left(g\sin\theta - \cos\theta\,\ddot{x}\right)$

$$= \frac{1}{l}\left[g\sin x_3 - \cos x_3 \frac{1}{(M+m) - m\cos^2 x_3}\left(m l x_4^2 \cdots\cdots\right)\right]$$

$$\Rightarrow \dot{x}_4 = \frac{g}{l}\sin x_3 - \frac{\cos x_3}{l\left(M+m - m\cos^2 x_3\right)}\left[m l x_4^2 \sin x_3 - mg\cos x_3 \sin x_3 + u\right]$$

So, $\dot{\mathbb{X}} = f(\mathbb{X}, u)$ is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{(M+m) - m\cos^2 x_3}\left[m l x_4^2 \sin x_3 - mg\cos x_3 \sin x_3 + u\right] \\ x_4 \\ \frac{g}{l}\sin x_3 - \frac{\cos x_3}{l\left(M+m - m\cos^2 x_3\right)}\left[m l x_4^2 \sin x_3 - mg\cos x_3 \sin x_3 + u\right] \end{bmatrix}$$

Ans.

Clearly not a linear model.

When $x_1 = x_0$, $x_2 = x_3 = x_4 = 0$ at some tim t,

$$f(\mathbb{X}, 0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow [x_0, 0, 0, 0]^T \text{ is an equilibrium point.}$$

<u>Linearization</u>

$\tilde{x}_1 = x_1 - x_0$

$\tilde{x}_2 = x_2 - 0 = x_2$

$\tilde{x}_3 = x_3 - 0 = x_3$

$\tilde{x}_4 = x_4 - 0 = x_4$

$$\Rightarrow \dot{\tilde{x}}_1 = \tilde{x}_2$$

$$\dot{\tilde{x}}_2 = \frac{1}{(M+m) - m\cos^2\tilde{x}_3}\left[ml\,\tilde{x}_4^2 \sin\tilde{x}_3 - mg\cos\tilde{x}_3\sin\tilde{x}_3 + u\right]$$

$$= \frac{}{\cancel{1+\frac{m}{M+m}}}\left[ml\,\tilde{x}_4^2\left(\tilde{x}_3 + O(\tilde{x}_3^2)\right) - mg\left(1 - \frac{\tilde{x}_3^2}{2} + \dots\right)\left(\tilde{x}_3 + \cdots\right) + u\right]$$

order $\tilde{x}_3^2$ terms

To find a linear approximation of $\frac{1}{M+m - m\cos^2\tilde{x}_3}$ for small $\tilde{x}_3$,

define $f(x) = \frac{1}{a + b\cos^2 x}$.

and ~~linearize~~ find Taylor series expansion of $f(x)$ around $x = 0$

$$f(x) = f(0) + \left.\frac{\partial f}{\partial x}\right|_{x=0} x + \cdots$$ (need only 1st term for linear approximation)

$$= \frac{1}{a+b} + \left.\frac{0 + 1(2b\cos x\sin x)}{(a+b\cos^2 x)^2}\right|_{x=0} \cdot x$$

$$= \frac{1}{a+b} + \frac{0}{(a+b)^2}\cdot x + \dots O(x^2)$$

$$\approx \frac{1}{a+b}$$ (linear approximation)

$$\Rightarrow \dot{\tilde{x}}_2 \approx \frac{1}{M+m-m}\left[0 - mg\tilde{x}_3 + u\right]$$ since $\tilde{x}_3, \tilde{x}_4$ etc are small

$$= -\frac{m}{M}g\tilde{x}_3 + \frac{u}{M}$$

$$\dot{\tilde{x}}_3 = \tilde{x}_4$$

$$\dot{\tilde{x}}_4 = \frac{g}{l}\sin\tilde{x}_3 - \frac{\cos\tilde{x}_3}{l\left(M+m-m\cos^2\tilde{x}_3\right)}\left[ml\tilde{x}_4^2\sin\tilde{x}_3 - mg\cos\tilde{x}_3\sin\tilde{x}_3 + u\right]$$

$$\approx \frac{g}{l}\tilde{x}_3 - \frac{1}{l.(M+m-m.1)}\cdot\left[0 - mg.\,1.\,\tilde{x}_3 + u\right]$$

$$= \frac{g}{l}\tilde{x}_3 + \frac{mg}{Ml}\tilde{x}_3 - \frac{1}{Ml}.u$$

$$= \frac{g}{l}\left(1+\frac{m}{M}\right)\tilde{x}_3 - \frac{1}{Ml}u$$

Collecting all these, we get

$$\begin{bmatrix}\dot{\tilde{x}}_1\\ \dot{\tilde{x}}_2\\ \dot{\tilde{x}}_3\\ \dot{\tilde{x}}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -\frac{m}{M}g & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & \frac{g}{l}\left(1+\frac{m}{M}\right) & 0\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\\ \tilde{x}_3\\ \tilde{x}_4\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{M}\\ 0\\ -\frac{1}{Ml}\end{bmatrix}u$$

~~y~~ since it is not specified what variables are measured, we dont have a well defined y.

(so, no $y = Cx$)