MIMO systems

Consider the following model of the suspension system of a car (see Problem , HW 1 for details)

$$m_b \ddot{y}_2 + b(\dot{y}_2 - \dot{y}_1) + k_s(y_2 - y_1) = u \quad \text{—— ①}$$

$$m_w \ddot{y}_1 + b(\dot{y}_1 - \dot{y}_2) + k_s(y_1 - y_2) + k_w y_1 = f - u \quad \text{—— ②}$$

For simplicity,

This is a system of coupled ODEs, ① cannot be solved for $y_2(t)$ without solving ① for $y_1(t)$ first, but $y_1(t)$ cannot be solved for without solving for $y_2(t)$ first. Later we will see that the solution of both $y_1(t)$ and $y_2(t)$ can be obtained concurrently by first expressing ① and ② in state space form.

For now, we want to obtain a tranfer function representation of ① and ②. Clearly, there are 2 inputs $u(t)$ and $f(t)$, and 2 outputs $y_1(t)$ and $y_2(t)$. Both u and f affect $y_1(t)$, as well as $y_2(t)$. Let $G_{y_1u}(s)$ be the T.F from u to y_1, ~~and~~ let $G_{y_2u}(s)$ be the T.F. from u to y_2.

Similarly, let $G_{y_1f}(s)$, $G_{y_2f}(s)$ be the T.F s from f to y_1, and from f to y_2, respectively. The question is, how do we find these transfer functions.

Upon taking Laplace transform of both sides of ① we get (assuming zero initial conditions for both y_1 and y_2)

$$(m_b s^2 + bs + k_s)\, Y_2(s) - (bs + k_s)\, Y_1(s) = U(s)$$

$$(m_w s^2 + bs + k_s + k_w)\, Y_1(s) - (bs + k_s)\, Y_2(s) = F(s) - U(s)$$

which can be expressed as:

$$\underbrace{\begin{bmatrix} -(bs+k_s) & m_b s^2 + bs + k_s \\ m_w s^2 + bs + k_s + k_w & -(bs+k_s) \end{bmatrix}}_{M(s)} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} U(s) \\ F(s) - U(s) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} U(s) \\ F(s) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = M^{-1}(s) \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} U \\ F \end{bmatrix}$$, assuming $M(s)$ is invertible.

$$M^{-1} = \frac{1}{\det(M)}\, \operatorname{adj}(M) \qquad M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

$$\det M = m_{11} m_{22} - m_{12} m_{21}$$

$$= (bs+k_s)^2 - (m_b s^2 + bs + k_s)(s^2 m_w + bs + k_s + k_w)$$

$$\operatorname{adj}(M) = \begin{bmatrix} -(bs+k_s) & -(m_w s^2 + bs + k_w + k_s) \\ -(m_b s^2 + bs + k_s) & -(bs+k_s) \end{bmatrix}^T$$

expanding the expression for the determinant, we get

$$\det(M) = -\left(m_b m_w s^4 + b(m_w + m_b)s^3 + \left(k_s(m_b + m_w) + k_w m_b\right)s^2 + b k_w s + k_w k_s \right)$$

$$\Rightarrow M^{-1} = \frac{1}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \begin{bmatrix} \dfrac{bs + k_s}{m_w m_b} & \dfrac{m_b s^2 + bs + k_s}{m_w m_b} \\ \\ \dfrac{m_w s^2 + bs + k_w + k_s}{m_w m_b} & \dfrac{bs + k_s}{m_b m_w} \end{bmatrix}$$

where
$$a_1 = \frac{b(m_w + m_b)}{m_b m_w}$$
$$a_2 = \frac{k_s(m_b + m_w) + k_w m_b}{m_b m_w}$$
$$a_3 = \frac{b k_w}{m_b m_w}$$
$$a_4 = \frac{k_s k_w}{m_b m_w}$$

Note: Whether SISO or MIMO, we always, always, always express the denominator as a <u>monic</u> polynomial (that is, a polynomial in which the coefficient of the highest power term is 1.)

Also define $\beta_1 = \frac{b}{m_w m_b}$, $\beta_2 = \frac{k_s}{m_w m_b}$, $\beta_3 = \frac{1}{m_w}$, $\beta_4 = \frac{1}{m_b}$

$\beta_5 = \frac{k_w + k_s}{m_w m_b}$, . Then

$$M^{-1} = \frac{1}{\underbrace{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}_{\text{call it } D(s)}} \begin{bmatrix} \beta_1 s + \beta_2 & \beta_3 s^2 + \beta_1 s + \beta_2 \\ \beta_4 s^2 + \beta_1 s + \beta_5 & \beta_1 s + \beta_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \frac{1}{D}\begin{bmatrix} \beta_1 s + \beta_2 & \beta_3 s^2 + \beta_1 s + \beta_2 \\ \beta_4 s^2 + \beta_1 s + \beta_3 & \beta_1 s + \beta_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} U \\ F \end{pmatrix}$$

$$= \frac{1}{D}\begin{bmatrix} -(\beta_3 s^2 + \beta_1 s + \beta_2) & \beta_3 s^2 + 2\beta_1 s + 2\beta_2 \\ -(\beta_1 s + \beta_2) & \beta_4 s^2 + 2\beta_1 s + \beta_3 + \beta_2 \end{bmatrix}\begin{pmatrix} U \\ F \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}\begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix}$$

where $G_{11}(s) = \dfrac{-(\beta_3 s^2 + \beta_1 s + \beta_2)}{D(s)}$, $\quad G_{12} = \dfrac{\beta_3 s^2 + 2\beta_1 s + 2\beta_2}{D(s)}$

$G_{21}(s) = \dfrac{-(\beta_1 s + \beta_2)}{D(s)} \qquad G_{22}(s) = \dfrac{\beta_4 s^2 + 2\beta_1 s + \beta_3 + \beta_2}{D(s)}$

and $U_1(s) = U(s)$

$U_2(s) = F(s)$

The reason for ~~writing~~ renaming U & F to U_1 & U_2 is that we can now say "$G_{ij}(s)$ is the transfer function from the j-th input to the i-th output".

each of the $G_{ij}(s)$ is a ratio of 2 polynomials, with a monic denominator, and it has a relative degree ≥ 1. So these are transfer functions we are familiar with.

These G_{12}, G_{21} etc are the transfer functions we set out to find. In fact, $G_{y_1 u}(s)$ is $G_{11}(s)$

Similarly, can you determine $G_{y_2 u} = ?$

$G_{y_1 f} = ?$

$G_{y_2 f} = ?$

Important point to note is that for a MIMO system, the transfer function G is a matrix, that consists of a number of SISO transfer functions.

We can write $$\mathbb{Y}(s) = \mathbb{G}(s)\,\mathbb{U}(s)$$

where $\mathbb{Y}(s) = \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix}$, $\mathbb{G} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}$,

and $\mathbb{U}(s) = \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$

In general, for a MIMO system with m inputs and p outputs, the T.F. $\mathbb{G}(s)$ is a $p \times m$ matrix, ~~of SISO~~ each of whose entries is a SISO T.F.

In particular, $G_{ij}(s)$, the (i,j)-th entry of $\mathbb{G}(s)$ is the transfer function from the j-th input to the i-th output

Just like for SISO systems, stability of MIMO systems can also be characterized in terms of the poles of the transfer function matrix $G(s)$. Since the <u>denominator of each of the SISO T.F.s that appear as entries of $G(s)$ is the same</u>, <u>the poles of the matrix T.F. $G(s)$ are simply the poles of any one of the SISO T.F. $G_{ij}(s)$ inside $G(s)$</u>

(this is a crucial point).

- zeros of $G(s)$: another business.

Looking back at the calculations we just did, poles of G are the roots of $\det(M(s)) = 0$

where $\begin{bmatrix} M(s) \end{bmatrix} \begin{bmatrix} Y \\ Y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} U \\ F \end{bmatrix}$

Later we will see an alternate characterization of the poles of a MIMO T.F.

— x —

Linear differential equation to state space representation

consider mass – spring – damper system :

$$m\ddot{y} + b\dot{y} + ky = u \quad \Rightarrow \quad \text{TF} \quad \frac{Y(s)}{U(s)} = \frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}$$

$\Downarrow$ S.S. (state space)

Second order D.E. with a single variable y

so we need 2 states !

Define $x_1 \triangleq y$

$x_2 \triangleq \dot{x}_1 \quad (= \dot{y} \; : \text{will need this later})$

Find $\dot{x}_1, \dot{x}_2$

$\dot{x}_1 = x_2 \quad$ (by defn)

$\dot{x}_2 = \ddot{x}_1 = \ddot{y} = -\frac{b}{m}\dot{y} - \frac{k}{m}y + u$

$= -\frac{b}{m}x_2 - \frac{k}{m}x_1 + u$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{b}{m}x_2 - \frac{k}{m}x_1 + u \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}_{\mathbb{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbb{B}} u$$

$\mathbb{X} \triangleq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$: state vector

$\mathbb{U} = u$: input vector

$\mathbb{Y} = y$: output vector

$$\Rightarrow \mathbb{Y} = y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbb{C}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{0}_{\mathbb{D}}.u$$

$$\Rightarrow \boxed{\begin{aligned} \dot{\mathbb{X}} &= \mathbb{A}\mathbb{X} + \mathbb{B}.\mathbb{U} \\ \mathbb{Y} &= \mathbb{C}\mathbb{X} + \mathbb{D}.\mathbb{U} \end{aligned}}$$

state space form

For an n-th order Linear D.E. involving a single variable y, we will need n- states.

General case:

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1}\dot{y} + a_n y = bu$$

TF: $$\frac{Y(s)}{U(s)} = \frac{b}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1}s + a_n}$$

SS: ① Define n- states, with ith state as the derivative of the ~~i-th~~ $(i-1)^{th}$ state, and $x_1 = y$.

$$x_1 \triangleq y$$
$$x_2 = \dot{x}_1$$
$$x_3 = \dot{x}_2$$
$$\vdots$$
$$x_{n-1} = \dot{x}_{n-2}$$
$$x_n = \dot{x}_{n-1}$$

② Find $\dot{x}_1 = ?,\ \dot{x}_2 = ?,\ \dot{x}_3 = ? \ldots\ \dot{x}_{n-1} = ?,\ \dot{x}_n = ?$

~~By def^n~~

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \end{aligned}\right\} \text{by def}^n$$

$$\dot{x}_n = \ddot{x}_{n-1} = \dddot{x}_{n-2} = \cdots = x_1^{(n)} = y^{(n)}$$

$$\Rightarrow \dot{x}_n = -a_1 y^{(n-1)} - a_2 y^{(n-2)} - \cdots - a_{n-1}\dot{y} - a_n y + bu$$

$$= -a_n x_1 - a_{n-1} x_2 - \cdots - a_2 x_{n-2} - a_1 x_{n-1} + bu$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ & & & & & 0 \\ 0 & \cdots & & & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_2 & -a_1 \end{bmatrix}}_{\mathbb{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}}_{\mathbb{X}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ b \end{bmatrix}}_{\mathbb{B}} \underbrace{u}_{\mathbb{U}}$$

~~When you have~~

$$y = [\underbrace{1\ 0\ 0 \cdots\cdots}_{\mathbb{C}}\ 0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \underbrace{0}_{\mathbb{D}.} \cdot u$$

When you have a system of coupled D.E involving multiple variables, the number of states needed to describe the system in state space form will depend on the number of variables as well as their highest order derivatives.

<u>Example:</u> $m_b \ddot{y}_2 + b(\dot{y}_2 - \dot{y}_1) + k_s(y_2 - y_1) = u$

$m_w \ddot{y}_1 + b(\dot{y}_1 - \dot{y}_2) + k_s(y_1 - y_2) = f - u$
$+ k_w y_1$

(suspension system of a car)

2 variables y_1, y_2, highest derivative : 2 (for both $y_1 \cdot y_2$)

① Define $x_1 = y_1$

$x_2 = \dot{x}_1 \qquad (= \dot{y}_1)$

$x_3 = y_2$

$x_4 = \dot{x}_3 \qquad (= \dot{y}_2)$

(2) Find $\dot{x}_1,\ x_2,\ \dot{x}_3,\ \dot{x}_4$

$$\dot{x}_1 = x_2 \quad (\text{by def}^n) \qquad \text{and} \qquad \dot{x}_3 = x_4 \quad (\text{by def}^n)$$

Now,
$$\dot{x}_2 = \ddot{x}_1 = \ddot{y}_1 = -\frac{b}{m_w}(\dot{y}_1 - \dot{y}_2) - \frac{k_s}{m_w}(y_1 - y_2) - \frac{k_w}{m_w} y_1 + f - u$$

$$= -\frac{b}{m_w}(x_2 - x_4) - \frac{k_s}{m_w}(x_1 - x_3) - \frac{k_w}{m_w} x_1 + f - u$$

$$= -\frac{k_s + k_w}{m_w} x_1 - \frac{b}{m_w} x_2 + \frac{k_s}{m_w} x_3 + \frac{b}{m_w} x_4 + f - u$$

and
$$\dot{x}_4 = \ddot{x}_3 = \ddot{y}_2 = -\frac{b}{m_b}(\dot{y}_2 - \dot{y}_1) - \frac{k_s}{m_b}(y_2 - y_1) + u$$

$$= -\frac{b}{m_b}(x_4 - x_2) - \frac{k_s}{m_b}(x_3 - x_1) + u$$

$$= \frac{k_s}{m_b} x_1 + \frac{b}{m_b} x_2 - \frac{k_s}{m_b} x_3 - \frac{b}{m_b} x_4 + u$$

Now that we have found expressions for $\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4$ in terms of the states and inputs,

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}}_{\dot{\mathbb{X}}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_s + k_w}{m_w} & -\frac{b}{m_w} & \frac{k_s}{m_w} & \frac{b}{m_w} \\ 0 & 0 & 1 & 0 \\ \frac{k_s}{m_b} & \frac{b}{m_b} & -\frac{k_s}{m_b} & -\frac{b}{m_b} \end{bmatrix}}_{\mathbb{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbb{X}} + \underbrace{\begin{bmatrix} 0 & 0 \\ -1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}}_{\mathbb{B}} \underbrace{\begin{bmatrix} u \\ f \end{bmatrix}}_{\mathbb{U}}$$

pay attention to $\mathbb{B}$ & $\mathbb{U}$: matrix and vector, since there are multiples inputs !

$\boxed{\text{Also, } \mathbb{C}, \mathbb{D}}$

$$\mathbb{Y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbb{C}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbb{D}} \begin{bmatrix} u \\ f \end{bmatrix}$$

If only y_1 were to be measured, then

$$Y = y_1 = [x_1] = \underbrace{[1 \quad 0 \quad 0 \quad 0]}_{\mathbb{C}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{[0 \quad 0]}_{\mathbb{D}} \begin{bmatrix} u \\ f \end{bmatrix}$$

So depending on what is the output, $\mathbb{C}$ and $\mathbb{D}$ may change.

Non-uniqueness of state space representation

Note that we could have chosen $x_1 = y_2$, $x_2 = \dot{x}_1$, $x_3 = y_1$, $x_4 = \dot{x}_3$, or even $x_1 = y_2$, $x_2 = y_1$, $x_3 = \dot{y}_2$, $x_4 = \dot{y}_1$.

In each case, the matrices $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$ would have been different.

For this reason, the state space representation of a dynamic system is <u>non-unique</u>.

$\boxed{\text{Sol}^n \text{ of } \dot{\mathbb{X}} = \mathbb{A}\mathbb{X} + \mathbb{B}\mathbb{U}}$

The solution $\mathbb{X}(t)$ of a system of differential equations expressed in state space form $\dot{\mathbb{X}}(t) = \mathbb{A}\,\mathbb{X}(t) + \mathbb{B}\,\mathbb{U}(t)$, is given by

$$\mathbb{X}(t) = e^{\mathbb{A}(t-t_0)}\,\mathbb{X}(t_0) + \int_{t_0}^{t} e^{\mathbb{A}(t-\tau)}\,\mathbb{B}\,\mathbb{U}(\tau)\,d\tau$$

where t_0 is any <u>initial</u> time, such that the initial condition $\mathbb{X}(t_0)$ is given, and $\mathbb{U}(t)$ is specified as a function of time.

[In the above, we are using "matrix exponential"

For any square matrix M, the matrix exponential e^M is defined as an infinite series

$$e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots + \frac{M^n}{n!} + \cdots$$

where I is the identity matrix of the same dimension as M]

Given $\mathbb{A}$, $\mathbb{B}$, $\mathbb{U}(t)$, $\mathbb{X}(t_0)$, $\mathbb{X}(t)$ can be computed from the formula above by numerical integration.

from t_0 to t

Thus, state space form is useful in numerically "solving" linear differential equations, especially when we have a number of coupled equations. (think: car suspension)

If the initial time t_0 is 0, we have

$$\mathbb{X}(t) = e^{\mathbb{A}t}\,\mathbb{X}(0) + \int_0^t e^{\mathbb{A}(t-\tau)}\,\mathbb{B}\,\mathbb{U}(\tau)\,d\tau$$

<u>Recap of notations</u>

$$\dot{\mathbb{X}} = \mathbb{A}\mathbb{X} + \mathbb{B}\mathbb{U}$$
$$\mathbb{Y} = \mathbb{C}\mathbb{X} + \mathbb{D}\mathbb{U}$$

n : # of states $\mathbb{X} = [x_1, x_2, \cdots x_n]^T$

m : # of inputs : $\mathbb{U} = [u_1, u_2, \cdots u_m]^T$

p : # of outputs : $\mathbb{Y} = [y_1, \cdots y_p]^T$

$\mathbb{A} = n \times n$ (<u>always</u> square) ~~real~~

$\mathbb{B} = n \times m$ (<u>almost always</u> tall & skinny) $(n > m)$

$\mathbb{C} = p \times n$ (almost alway fat) $(p < n)$

$\mathbb{D} = p \times m$

all the entries of $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$ are real numbers.

$\mathbb{A}$: dynamics matrix

$\mathbb{B}$: input "

$\mathbb{C}$: output "

$\mathbb{D}$: feedthrough " ($\mathbb{D}$ is frequently a 0 matrix)

ex: $n = 4$, $m = 2$, $p = 3$

$$\overset{\dot{\mathbb{X}}}{\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}} = \overset{\mathbb{A}}{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}} \overset{\mathbb{X}}{\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}} + \overset{\mathbb{B}}{\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}} \overset{\mathbb{U}}{\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}}$$

$$\underset{\mathbb{Y}}{\begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}} = \underset{\mathbb{C}}{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}} \underset{\mathbb{X}}{\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}} + \underset{\mathbb{D}}{\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}} \underset{\mathbb{U}}{\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}}$$