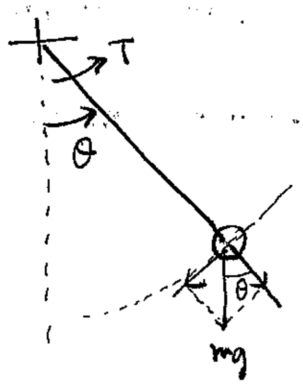
Non-linear dynamic systems
and their state-space representation

$$\dot{\mathbb{X}} = f(\mathbb{X}, \mathbb{U}, t).$$

Example: pendulum tip mass : m
effective length : l



counter clockwise (ccw) : positive

$T(t)$: external moment in +ve direction

air drag = proportional to linear velocity

Force balance leads to

$$(ml^2)\,\ddot{\theta} = T + (-mg\sin\theta)\,l + (-b\dot{\theta}l)\,l$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{l}\sin\theta - \frac{b}{m}\dot{\theta} + \frac{1}{ml^2}T(t)$$

$$= -\frac{b}{m}\dot{\theta} - \frac{g}{l}\sin\theta + u(t) \quad \text{— (1)} \qquad \left\{ u(t) \triangleq \frac{T(t)}{ml^2} \right\}$$

To express this differential equation in state-space form, define $x_1 = \theta$

$x_2 \triangleq \dot{x}_1 = (\dot{\theta})$

[Note that (1) is a non-linear ODE]

and then find equations for $\dot{x}_1$ and $\dot{x}_2$:

$$\dot{x}_1 = x_2 \quad (\text{by def}^n) \quad \text{— (2)}$$

$$\dot{x}_2 = \ddot{x}_1 = \ddot{\theta} = -\frac{b}{m}\dot{\theta} - \frac{g}{l}\sin\theta + u$$

$$= -\frac{b}{m}x_2 - \frac{g}{l}\sin x_1 + u \quad \text{— (3)}$$

Because $\sin x_1$ is a non-linear function of the state x_1, these 2 equations cannot be expressed as $\dot{X} = AX + Bu$.

Instead, define
$$\left.\begin{aligned} f_1(x_1, x_2, u) &\triangleq x_2 \\ f_2(x_1, x_2, u) &\triangleq -\frac{b}{m}x_2 - \frac{g}{l}\sin x_1 + u \end{aligned}\right] \quad (4)$$

where $f_1(\ )$ and $f_2(\ )$ are ~~possibly~~ non linear (in general) functions of the states x_1, x_2 and input u.

Now we write (2) and (3) as
$$\dot{x}_1 = f_1(x_1, x_2, u) = f_1(\mathbf{x}, u)$$
$$\dot{x}_2 = f_2(x_1, x_2, u) = f_2(\mathbf{x}, u)$$

f_1, f_2 are to be thought of as functions of the state vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and input u.

Define a <u>vector function</u> $\mathbf{f}$ as:
$$\mathbf{f}(\mathbf{x}, u) = \begin{bmatrix} f_1(\mathbf{x}, u) \\ f_2(\mathbf{x}, u) \end{bmatrix} \quad \text{— (5)}$$

then we have
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u) \quad \text{— (6)}$$

In general, any number of coupled ODEs can be expressed as

$$\dot{\mathbb{X}} = \mathbb{f}(\mathbb{X}, \mathbb{U}, t) \quad \text{— (7)}$$

where the state vector $\mathbb{X} = [x_1, x_2, \ldots, x_n]^T$ has n scalar states, $\mathbb{U}$ is a vector of inputs (if there are more than one). The number of states, n, will depend on the number of ODEs that describe the system and the order of the derivatives that appear in them.

Notice that the general form (7) has a 't' in the right hand side. That can happen if the functions $f_i(\ )$ depend <u>explicitly</u> on time t.

For example, suppose the length l of the pendulum is not a constant, but ~~is~~ changes with time in a fixed manner. say $l = l_0 + a \sin(\omega_0 t)$, so that the length changes periodically with time, with a period $\frac{2\pi}{\omega}$ sec.

~~~~ <u>ex:</u> Child on a swing who moves her body periodically to change the effective length

<u>ex:</u> an aircraft with a flexible wing (?)
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In that case, equations (2) and (3) become

$$\left.\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b}{m}x_2 - \frac{g}{l_0 + a\sin\omega_0 t}\sin x_1 + u\end{aligned}\right]\quad (8)$$

the right hand side now depends explicitly on t

because of $\dfrac{g}{l_0 + a\sin\omega_0 t}$.

So, we have to redefine $f_1(\mathbb{X}, u, t) \triangleq x_2$

$$f_2(\mathbb{X}, u, t) = -\frac{b}{m}x_2 - \frac{g}{l_0 + a\sin\omega_0 t}\sin x_1 + u$$

so that (8) can be expressed as

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} f_1(\mathbb{X}, u, t) \\ f_2(\mathbb{X}, u, t)\end{bmatrix}$$

which is in the general form (7).

Systems in the form (7) are called <u>time varying</u> systems.

<u>Linear systems</u> ~~If (7) can be written~~

If a dynamic system (that is, a set of coupled ODEs that are expressed in the form (7))

can be written as

$$\dot{\mathbb{X}}(t) = \mathbb{A}(t)\,\mathbb{X}(t) + \mathbb{B}(t)\,\mathbb{U}(t), \quad \text{——} \ (9)$$

it is called a <u>linear</u> system.

If, furthermore, $\mathbb{A}(t)$ and $\mathbb{B}(t)$ are constant matrices, that do depend on time, so that (9) is, in fact,

$$\dot{\mathbb{X}}(t) = \mathbb{A}\,\mathbb{X}(t) + \mathbb{B}\,\mathbb{U}(t), \quad \text{— (10)}$$

then the system is called a Linear Time Invariant (LTI) system. ▨

(9) is called a Linear time Varying (LTV) system.

$\boxed{\text{Equilibrium point and linearization}}$:

(note: we'll only deal with $\dot{\mathbb{X}} = \mathbb{f}(\mathbb{X}, \mathbb{U})$ in this course)

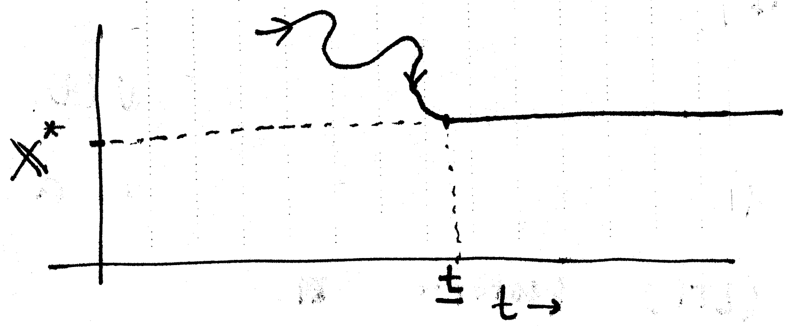
Consider a dynamic system [redacted]: all inputs are 0 for all time.

$$\dot{\mathbb{X}} = \mathbb{f}(\mathbb{X}, \mathbf{0}) \quad \text{— (11)}$$

An equilibrium point of the system (11) is a value of the state, say $\mathbb{X}^*$, so that if $\mathbb{X}(\underline{t}) = \mathbb{X}^*$ (for some time $\underline{t}$), then $\mathbb{X}(t) = \mathbb{X}^*$ for all time $t \geq \underline{t}$.

That is, if the state is at an equilibrium point at some time, it does not move from there any more. A moment's thought will reveal that this is possible only if $\mathbb{f}(\mathbb{X}^*, 0) = 0$

An abstract representation of what happens when the state of a system reaches an equilibrium point :



<u>Note:</u> The word point may be confusing. An equilibrium 'point' is still a vector. If the state vector X has n scalar entries $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ then X can be thought of a "point" in a n-dimensional space (we say $X \in \mathbb{R}^n$). Arguably it is a 'point' that keeps moving in the space as t changes. If the state reaches an equilibrium point at some time, it stays there for all future time. ■

<u>Example:</u> pendulum again !

To find the equilibrium point, set

$$f_1(X, 0) = 0 \quad \Rightarrow \quad x_2 = 0$$

$$f_2(X, 0) = 0 \quad \Rightarrow -\frac{b}{m} x_2 - \frac{g}{l} \sin x_1 + 0 = 0$$

$$\Rightarrow \sin x_1 = 0 \Rightarrow x_1 = 0 \text{ or } \pi$$

So, for the dynamic system ⑥ that describes a pendulum, we have 2 equilibrium points:

$$\mathbb{X}^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbb{X}^* = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

Linearization (around an equilibrium point)

"In general, define $\tilde{\mathbb{X}} \triangleq \mathbb{X} - \mathbb{X}^*$, and then derive Linear equations for $\dot{\tilde{\mathbb{X}}}$, assuming that $\tilde{\mathbb{X}}$ is small."

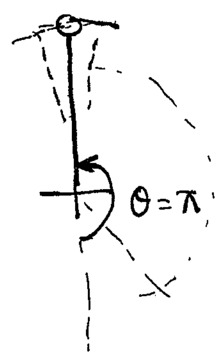
This procedure is called linearization around $\mathbb{X}^*$.

The justification comes from certain mathematical results that say that when $\mathbb{X}(t)$ is close to $\mathbb{X}^*$, (but not equal to $\mathbb{X}^*$), the behavior of the non-linear system is very similar to the behavior of the approximate, linearized system.

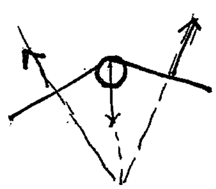
Ex: pendulum again!

Let us linearize ⑥ around the equilibrium point

$$\mathbb{X}^* = \begin{bmatrix} \pi \\ 0 \end{bmatrix} \leftarrow \text{(vertically upright position)}$$



This is called the inverted pendulum.

(fighter jets: )

define $\tilde{x}_1 = x_1 - x_1^* = x_1 - \pi$

$\tilde{x}_2 = x_2 - x_2^* = x_2$

assuming $\tilde{x}_1, \tilde{x}_2$ are small (meaning, $|\tilde{x}_1| << 1$, $|\tilde{x}_2| << 1$),

find linear approximations of $f_1(x_1, x_2, u)$ and $f_2(x_1, x_2, u)$

around $x_1 = \pi$ and $x_2 = 0$. (usually by Taylor series)

Note that we set $u=0$ only in finding the equilibrium points. After finding the equilibrium point, we linearize the original dynamic equations around it, while keeping the external signals.

$$\dot{x}_1 = f_1(x_1, x_2, u) = x_2 \Rightarrow \dot{(\tilde{x}_1 + \pi)} = \tilde{x}_2 + 0 \Rightarrow \boxed{\dot{\tilde{x}}_1 = \tilde{x}_2} \quad (12)$$

$$\dot{x}_2 = f_2(x_1, x_2, u)$$

$$\Rightarrow \dot{(\tilde{x}_2 + 0)} = -\frac{g}{l}\sin(\tilde{x}_1 + \pi) - \frac{b}{m}(\tilde{x}_2 + 0) + u$$

$$\Rightarrow \dot{\tilde{x}}_2 = +\frac{g}{l}\sin\tilde{x}_1 - \frac{b}{m}\tilde{x}_2 + u$$

Now use our assumption that $\tilde{x}_1$ is small.

$$\Rightarrow \sin\tilde{x}_1 \approx \tilde{x}_1$$

$$\boxed{\therefore \dot{\tilde{x}}_2 \approx \frac{g}{l}\tilde{x}_1 - \frac{b}{m}\tilde{x}_2 + u} \quad : \text{linear approximation!} \quad (13)$$

(12) and (13) give us

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad : \text{LTI system!}$$

Note that we write $\dot{x} = \cdots$, but the equality is to be understood as an approximation.