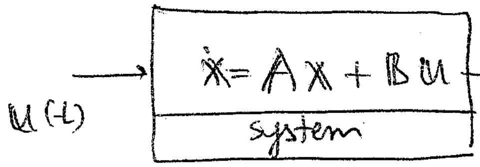
State feedback control

The entire state vector $x(t)$ can be measured, and therefore used for feedback



: open loop.

state feedback stabilization: control input $u(t)$ is a linear function of the state $x(t)$:

$u = -\mathbb{K}x$, where $\mathbb{K}$ is a matrix of appropriate dimension, so that the closed loop is stable,

Closed loop:

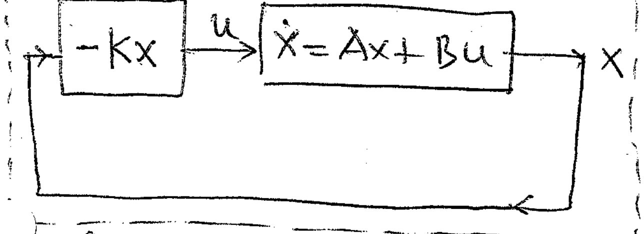
$$\left.\begin{array}{l} \dot{x} = Ax + Bu \\ u = -\mathbb{K}.x \end{array}\right\} \Rightarrow \dot{x} = Ax + \mathbb{B}(-Kx)$$

$$= (A - BK)\, x$$

$$\Rightarrow \dot{x} = A_{cL}\, x$$

where $A_{cL} = A - BK$

Closed loop



$\equiv$ $\boxed{\dot{x} = (A - BK)\, x}$

(all signals in this block diagram are in time domain)

Clearly, for stability of the closed loop system, we need $Re\ \lambda_i(A - \mathbb{B}\mathbb{K}) < 0 \quad i = 1, 2, \cdots n$

The stabilization problem is important when the open loop system $\dot{\mathbb{X}} = A\mathbb{X}$ is unstable.

(ex: inverted pendulum)

We then use feedback control to stabilize the closed loop.

It is also important when the open loop is stable, but its eigenvalues (poles) are such that the resulting time response of the open loop system is not satisfactory (think rise time, peak over shoot!)

In that case we can use state feedback to place the eigenvalues of $A - \mathbb{B}\mathbb{K}$ in that part of the complex plane so that the time response of the closed loop system $\dot{\mathbb{X}} = (A - \mathbb{B}\mathbb{K})\mathbb{X}$ is satisfactory.

In both cases, we have choose the gain matrix $\mathbb{K}$, so that these objectives are achieved.

<u>State feedback control design problem:</u> given an open loop system $\dot{\mathbb{X}} = \mathbb{A}\mathbb{X} + \mathbb{B}\mathbb{U}$ (meaning the matrices $\mathbb{A}_{n\times n}$ and $\mathbb{B}_{n\times m}$ are known), <u>and</u> given the desired eigenvalues $\lambda_1^{(d)}, \lambda_2^{(d)}, \ldots \lambda_n^{(d)}$, choose a $m\times n$ gain matrix $\mathbb{K}$, so that the eigenvalues of $(\mathbb{A} - \mathbb{B}\mathbb{K})$ are equal to the desired eigenvalues.

Before we can design such a gain $\mathbb{K}$, we have to study the more important question of <u>when</u> can this problem be solved? Meaning, given an <u>arbitrary</u> $\mathbb{A}, \mathbb{B}, \lambda_1^{(d)}, \ldots \lambda_n^{(d)}$, can one always find a gain $\mathbb{K}$ to satisfy the condition that

$$\lambda_i(\mathbb{A} - \mathbb{B}\mathbb{K}) = \lambda_i^{(d)} \quad , i = 1, 2, \ldots n \quad ?$$

<u>Ans</u>: not always. It is possible only when "$(\mathbb{A}, \mathbb{B})$ is completely controllable".

What does this mean?

<u>Definition</u>: Given a system $\dot{\mathbb{X}} = \mathbb{A}\mathbb{X} + \mathbb{B}\mathbb{U}$, we say that the system, or more precisely, $(\mathbb{A}, \mathbb{B})$,

is <u>completely controllable</u> if it is possible to take the state from an arbitrarily chosen point x_0 to another arbitrary point x_1 in finite time, say T, by using some control signal $\{u(t)\}_{t=0}^{t=T}$

How do you verify if (A, B) is completely controllable ?

<u>Define:</u> $$P_c \triangleq \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}_{n \times nm}$$

(where $n = \#$ of states)

P_c is called the <u>controllability matrix</u> of (A, B), or the controllability matrix of the system $\dot{x} = Ax + Bu$.

$$\boxed{\text{Result: } (A, B) \text{ is completely controllable if and only if } P_c \text{ has rank } n.}$$

So, if (A, B) is completely controllable, which can be checked by computing the rank of P_c, ~~the~~ it is possible to solve the state feedback control design problem stated earlier.

How to find $\mathbb{K}$ to solve the state feedback control design problem when (A, B) is completely controllable?

<u>Ans:</u> <u>several methods.</u>

For example, the method described in "Robust Pole assignment in linear state feedback" by J. Kautsky and N. K. Nichols, in International journal of Control, 41 (1985), pages 1129–1155, can be used.

This method has been implemented in Matlab, and can be called by the "place" command.

Alternatively for systems with small state space dimension, we can use <u>Ackerman's formula</u> (which is described next) to design $\mathbb{K}$ if the system has a <u>single input</u>

Suppose $\dot{x} = Ax + Bu$, with u a scalar input, is an n-th order system, and let $\lambda_1^{(d)}, \lambda_2^{(d)}, \ldots, \lambda_n^{(d)}$ be the desired closed loop eigenvalues.

Define the <u>desired</u> closed loop characteristic equation

$$q(s) \triangleq (s-\lambda_1^{(d)})(s-\lambda_2^{(d)})(\cdots)\cdots(s-\lambda_n^{(d)})$$

$$= s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1} s + \alpha_n$$

Define the matrix $q(A) \triangleq A^n + \alpha_1 A^{n-1} + \cdots + \alpha_{n-1} A + \alpha_n I.$

Let $\mathbb{P}_c = [\mathbb{B} \ \mathbb{A}\mathbb{B} \ \mathbb{A}^2\mathbb{B} \ \cdots \ \mathbb{A}^{n-1}\mathbb{B}]$ be the controllability matrix of $(\mathbb{A}, \mathbb{B})$.

[$(\mathbb{A}, \mathbb{B})$ is assumed completely controllable, which guarantees that $\mathbb{P}_c$ is invertible]

then, the gain $\mathbb{K}$ can be chosen as

$$\mathbb{K} = [0 \ 0 \cdots \ 1] \ \mathbb{P}_c^{-1} \ q(\mathbb{A}) \quad \text{———} \ (*)$$

which ensures that the eigenvalues of

$$(\mathbb{A} - \mathbb{B}.\mathbb{K})$$

are the desired ones: $\lambda_1^{(d)}, \lambda_2^{(d)}, \cdots \lambda_n^{(d)}$. The formula $(*)$ is called Ackerman's formula.

Ex: consider the plant : $\frac{Y(s)}{U(s)} = \frac{1}{s^2}$

(model of a satellite)

which has the state space representation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\mathbb{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbb{B}} u$$

with $x_1(t) = y(t)$, $x_2(t) = \frac{d}{dt} x_1(t)$.

Suppose we have both position x_1 and velocity x_2 available, and want to employ state feedback control to place the eigenvalues of

$(\mathbb{A} - \mathbb{B}.\mathbb{K})$ as $-1 \pm j$

Which means, $\lambda_1^{(d)} = -1 + j$

$\lambda_2^{(d)} = -1 - j$

(it does not matter even if we choose $\lambda_2^{(d)} = -1 + j$ and $\lambda_1^{(d)} = -1 - j$)

$$\Rightarrow q(s) = \left(s - (-1+j)\right)\left(s - (-1-j)\right)$$

$$= (s+1-j)(s+1+j)$$

$$= s^2 + 2s + 2$$

Since $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$q(A) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 + 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$$

Next, we need P_c^{-1} to apply Ackerman's formula.

$$P_c = [B \;\; AB] = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \;\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P_c^{-1} = \frac{1}{\det(P_c)} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Apply Ackerman's formula to choose the gain as

$$K = [0 \;\; 1]\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} = [1 \;\; 0]\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} = [2 \;\; 2]$$

Ans.

Check: Does this gain really place the closed loop eigenvalues at $-1 \pm j$?

$$A - BK = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}[2 \;\; 2] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

eigenvalues of $A - \mathbb{B}\mathbb{K}$ are roots of $\det(A - \mathbb{B}\mathbb{K} - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} -\lambda & 1 \\ -2 & -2-\lambda \end{bmatrix}\right)$$

$$= \lambda(2+\lambda) + 2$$

$$= \lambda^2 + 2\lambda + 2$$

~~root~~ $\Rightarrow$ eigenvalues of $A - \mathbb{B}\mathbb{K}$ are roots of $\lambda^2 + 2\lambda + 2 = 0$,

which are $-1 \pm j$. !

verified !

⊙ summary of state feedback

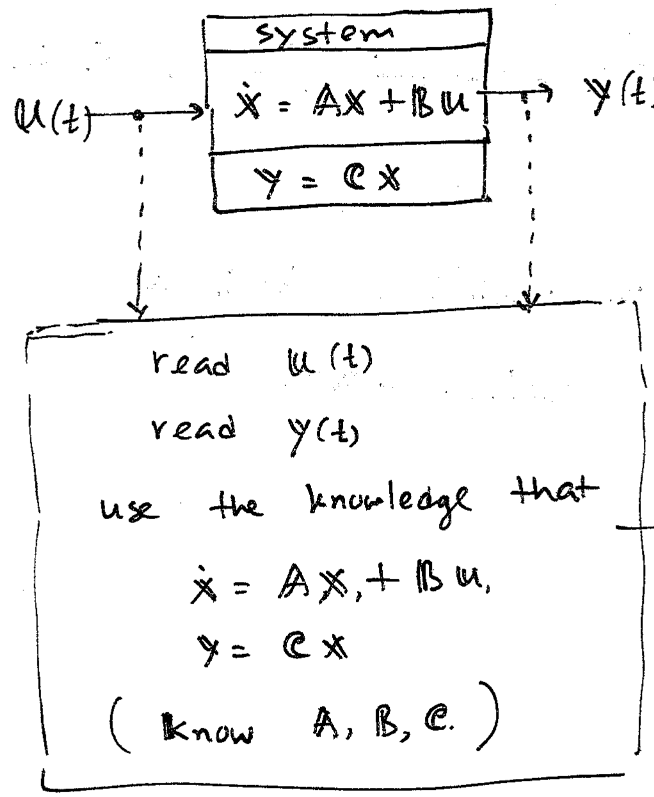
1. Check if $x(t)$ can be measured
2. Check if $(A, \mathbb{B})$ is completely controllable
3. If no, stop !

   if yes, A) choose desired closed loop eigenvalues and design $\mathbb{K}$ such that the eigenvalues of $(A - \mathbb{B}\mathbb{K})$ are at those locations.

   A.2 : To do this design, either use MATLAB ("place"), or, for single input systems, Ackerman's formula.

<u>Output feedback</u> : In practice, it is more common that the output $\mathbf{y}(t)$ available for feedback is not the whole state $\mathbf{x}(t)$, but a small part of it. In such a case, state feedback is not possible. What do we do then ?

In many such cases, it is possible to <u>reconstruct</u> the state from the knowledge of the inputs going into the system and outputs coming out of the system :



$\rightarrow$ figure out $\mathbf{x}(t)$ from all this information

When this is the case, we reconstruct the state $\mathbf{x}(t)$, ~~the~~ and use the reconstructed value ~~in pla~~ for state feedback.

Obvious question(s): Q1) What exactly is this reconstruction ?

Q2) When can it be done ?

Q3) How to do it ?

A1 : Definition of reconstructability :

A system $\dot{\mathbb{X}} = \mathbb{A}\mathbb{X} + \mathbb{B}\mathbb{U}$, $\mathbb{Y} = \mathbb{C}\mathbb{X}$ is said to be reconstructible, (or more precisely, $(\mathbb{A}, \mathbb{C})$ is said to be reconstructible) if there is a finite number T so that from the knowledge of $\mathbb{U}(t)$ and $\mathbb{Y}(t)$ during the interval $0 \leq t \leq T$, it is possible to determine $\mathbb{X}(t)$ ~~during the same~~ for all t in the internal $0 \leq t \leq T$.

A2 : When is $(\mathbb{A}, \mathbb{C})$ reconstructible ? It is when the rank of the observability matrix

$$\mathbb{P}_o \triangleq \begin{bmatrix} \mathbb{C} \\ \mathbb{C}\mathbb{A} \\ \mathbb{C}\mathbb{A}^2 \\ \vdots \\ \mathbb{C}\mathbb{A}^{n-1} \end{bmatrix}$$

is equal to n. (remember that n is the dimension of the state vector $\mathbb{X}$)

Ans to Q3 State reconstruction is done by designing something called an observer, which is a system (differential equation) whose state is designed to converge to the true state of the original system we are trying to reconstruct. The state of the new system (which can be computed by numerically solving it in a computer) serves as an estimate of the state we are trying to reconstruct.

(This estimate is used in place of the true state in the state feedback controller)

Details: Define a dynamic system (with state $\hat{x}$)

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \quad \text{—— (0)}$$

[A, B, C : of the original system $\dot{x} = Ax + Bu$, $y = Cx$, whose state x is unknown. The y and u in (0) are the outputs and inputs of the original system.]

$\underbrace{\hphantom{outputs and inputs}}_{\text{very important}}$

We will choose L (a matrix of appropriate dimension) so that $\hat{x}(t) \longrightarrow x(t)$ as $t \to \infty$.

How to do that?

First, define $e(t) = x(t) - \hat{x}(t)$ (error between the true state and the estimate)

$$\Rightarrow \dot{e} = \dot{x} - \dot{\hat{x}}(t)$$

We know
$$\dot{x} = Ax + Bu$$
$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$
$$= A\hat{x} + Bu + LC(x - \hat{x})$$
since $y = Cx$

Subtracting one from the other,

$$\dot{x} - \dot{\hat{x}} = A(x - \hat{x}) + 0 - LC(x - \hat{x})$$

$$\Rightarrow \dot{e} = Ae - LC.e$$

$$\Rightarrow \dot{e} = (A - LC)e$$

Clearly, $e(t) \to 0$ as $t \to \infty$ if and only if $Re\left(\lambda_i(A - LC)\right) < 0$, $i = 1, 2, \cdots n$.

If we choose L such that this condition is satisfied, then the estimation error $e(t)$ will go to 0.

So we can simply use $\hat{x}(t)$ in place of $x(t)$ in the state feedback control law $u = -Kx$, which leads to $\mathbb{U} = -\mathbb{K}\hat{\mathbb{X}}$ as the control signal sent into the system.

Designing $\mathbb{L}$: ① decide on desired eigenvalues of $\mathbb{A}-\mathbb{L}\mathbb{C}$.

call them $\lambda_1^{(d)}, \ldots, \lambda_n^{(d)}$.

② Notice that the eigenvalues of a matrix are the same as those of its transpose.

$\Rightarrow$ the eigenvalues of $\mathbb{A}^T - \mathbb{C}^T\mathbb{L}^T$ need to be at $\lambda_1^{(d)}, \ldots, \lambda_n^{(d)}$.

③ Solve for $\mathbb{L}^T$ by using Ackerman's formula, by replacing $\mathbb{A}$ by $\mathbb{A}^T$ and $\mathbb{B}$ by $\mathbb{C}^T$.

(assuming $\mathbb{C}^T$ is a single column, that is, we have a single output).

In particular, $q(\mathbb{A})$ will be replaced by $q(\mathbb{A}^T)$ and

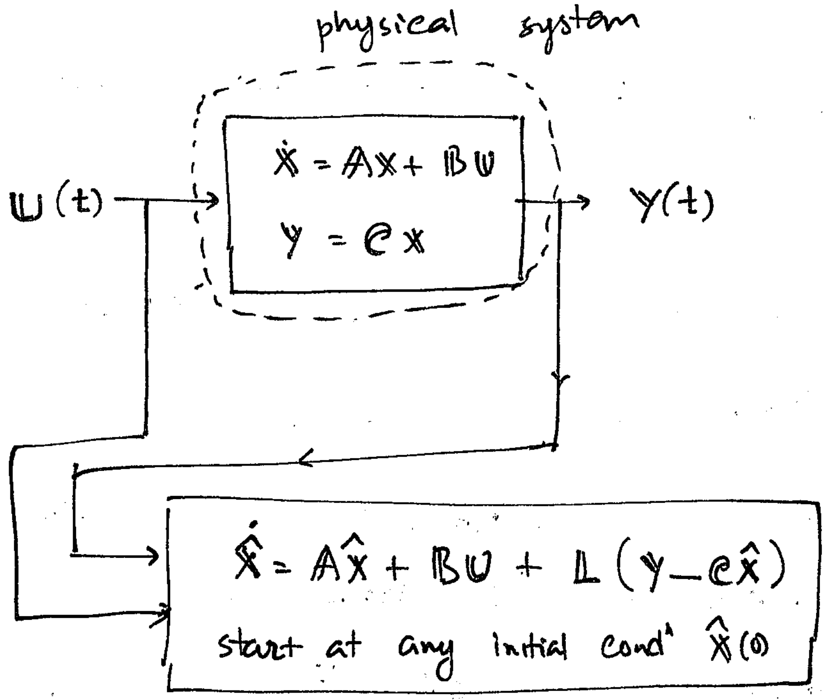
$\mathbb{P}_c = [\mathbb{B} \;\; \mathbb{A}\mathbb{B} \cdots \mathbb{A}^{n-1}\mathbb{B}]$ will be replaced by

$[\mathbb{C}^T \;\; \mathbb{A}^T\mathbb{C}^T \cdots (\mathbb{A}^T)^{n-1}\mathbb{C}^T]$, which is $\mathbb{P}_o^T$

so, $\mathbb{L}^T = [0 \; 0 \cdots 1](\mathbb{P}_o^T)^{-1} q(\mathbb{A}^T)$

$$\Rightarrow \mathbb{L} = (\quad " \quad)^T = [q(\mathbb{A}^T)]^T \mathbb{P}_o^{-1} \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$= q(\mathbb{A})\, \mathbb{P}_o^{-1} \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$$



solve this in a computer in real time.

observer

Summary of output feedback through observer design

1. Check if (A, C) is completely reconstructible.
2. If not, stop!

   if yes, design observer to estimate x

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

To design L, decide on desired eigenvalues of $A - LC$, and then use either MATLAB or, in case of a single output system, Ackerman's formula, so that eigenvalues of $A - LC$ are in those locations.