Name: :UFID

Problem 1. [10 points]

Sketch the Bode plot of the following transfer function:

$$G(s) = \frac{2}{s^2 + 22s + 40}$$

Label the x- and y-axis of both the magnitude and phase plots carefully.

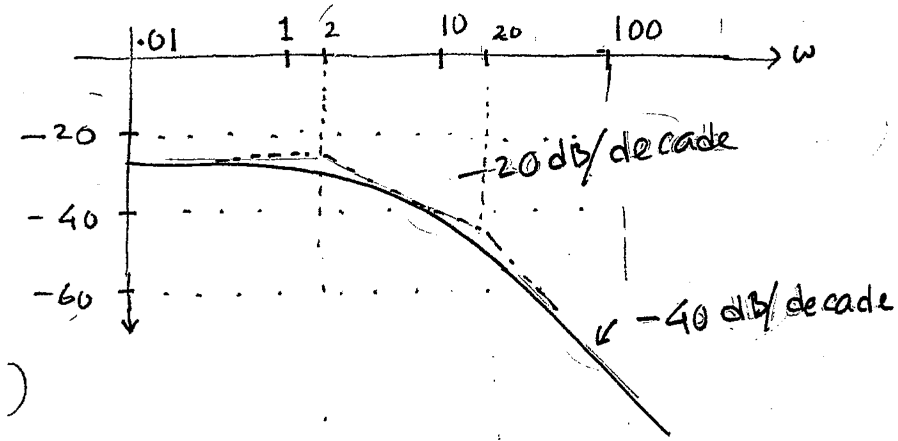
$$G(s) = \frac{2}{(s+2)(s+20)} \Rightarrow G(j\omega) = \frac{2}{(j\omega+2)(j\omega+20)} = \frac{\frac{1}{20}}{\left(j\frac{\omega}{2}+1\right)\left(j\frac{\omega}{20}+1\right)}$$

$$20\log_{10}|G(j\omega)| = 20\log_{10}\frac{1}{20} - 20\log_{10}\left|\left(j\frac{\omega}{2}+1\right)\right| - 20\log_{10}\left|j\frac{\omega}{20}+1\right|$$

$$= -26.02 - 20\log_{10}\left|j\frac{\omega}{2}+1\right| - 20\log_{10}\left|\frac{j\omega}{20}+1\right|$$

↑ breakpoint at $w=2$ ↑ breakpoint at $\omega = 20$

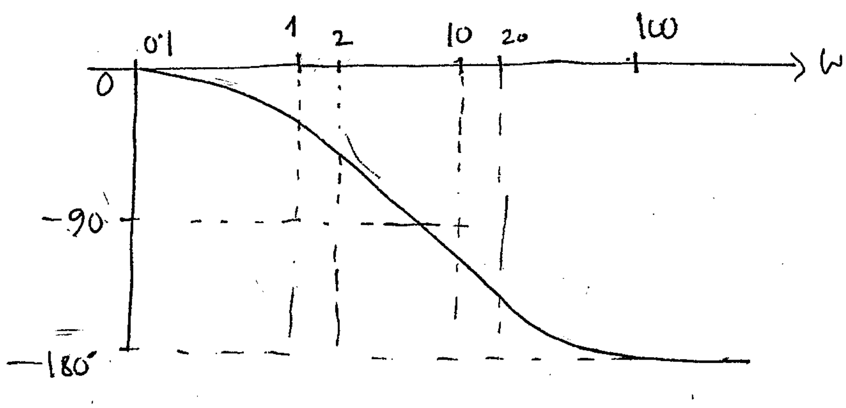
$|G(j\omega)|$ (dB)



$\angle G(j\omega) = 0 - \angle\left(j\frac{\omega}{2}+1\right) - \angle\left(\frac{j\omega}{20}+1\right)$

total phase change $= -180°$

$\angle G(j\omega)$ (degree)



Problem 2. [10 points]

Suppose the system $H(s) = \frac{1}{s+1}$ is driven by the input signal $x(t) = \cos(5t) + 10\sin(10t + \pi/3)$. What will be the output signal $y(t)$ at steady state?

$\cos 5t \longrightarrow \boxed{\frac{1}{s+1}} \longrightarrow$ steady state output $= \left|\frac{1}{j5+1}\right| \cos\left(5t + \angle\left(\frac{1}{j5+1}\right)\right)$

$= \frac{1}{\sqrt{26}} \cos(5t \cancel{+} - 1.373)$

$10\sin\left(10t + \frac{\pi}{3}\right) \longrightarrow \boxed{\frac{1}{s+1}} \longrightarrow$ steady state output $=$

$= 10 \left|\frac{1}{j10+1}\right| \sin\left(10t + \frac{\pi}{3} + \angle\left(\frac{1}{j10+1}\right)\right)$

$= \frac{10}{\sqrt{101}} \sin\left(10t + \frac{\pi}{3} - 1.4711\right)$

$\Rightarrow x(t) = \cos 5t + 10\sin\left(10t + \frac{\pi}{3}\right) \rightarrow \boxed{\frac{1}{s+1}} \rightarrow$ steady state output

$= \underset{0.1961}{\frac{1}{\sqrt{26}}} \cos(5t \cancel{+} - \underset{78.69}{1.373}) + \underset{0.995}{\frac{10}{\sqrt{101}}} \sin(10t - \underset{24.29}{0.4239})$

Problem 3. [5 + 5 = 10 points]

1. What is the natural frequency and damping ratio of $G(s) = \dfrac{s + 20}{2s^2 + 6s + 20}$?

2. Let $G_1(s) = \dfrac{k}{s^2 + 2s + 10}$ and $G_2(s) = \dfrac{s + 5}{(s+0.02s + 0.5)(s^2 + 2s + 50)}$. Determine the value of k so that the steady state outputs of $G_1(s)$ and $G_2(s)$ in response to a unit step input are the same?

1. $G(s) = \dfrac{\frac{1}{2}(s+20)}{s^2 + 3s + 10} \Rightarrow \omega_n^2 = 10 \Rightarrow \omega_n = \sqrt{10} = 3.162$ and

$2\zeta\omega_n = 3 \qquad \zeta = \dfrac{3}{2\sqrt{10}} = 0.4743$

2. steady state output of $G_1(s)$ in response to unit step input

is $1 \cdot G_1(0) = \dfrac{k}{10}$

steady state output of $G_2(s)$ . . . . is $1 \cdot G_2(0) = \dfrac{5}{.5 \cdot 50} = \dfrac{1}{5}$

$\therefore$ we need $\dfrac{k}{10} = \dfrac{1}{5} \Rightarrow \underline{k = 2}$ Ans

Problem 4. [10 pt]

A second order transfer function $H(s)$ has the following transient response specifications: rise time $t_r \leq 0.1$ sec, peak overshoot $M_p \leq 0.16$, and settling time $t_s \leq 0.511$ sec. Sketch (accurately) the region in the complex plane where the poles of $H(s)$ may lie to meet these specifications.

second order : apply formula

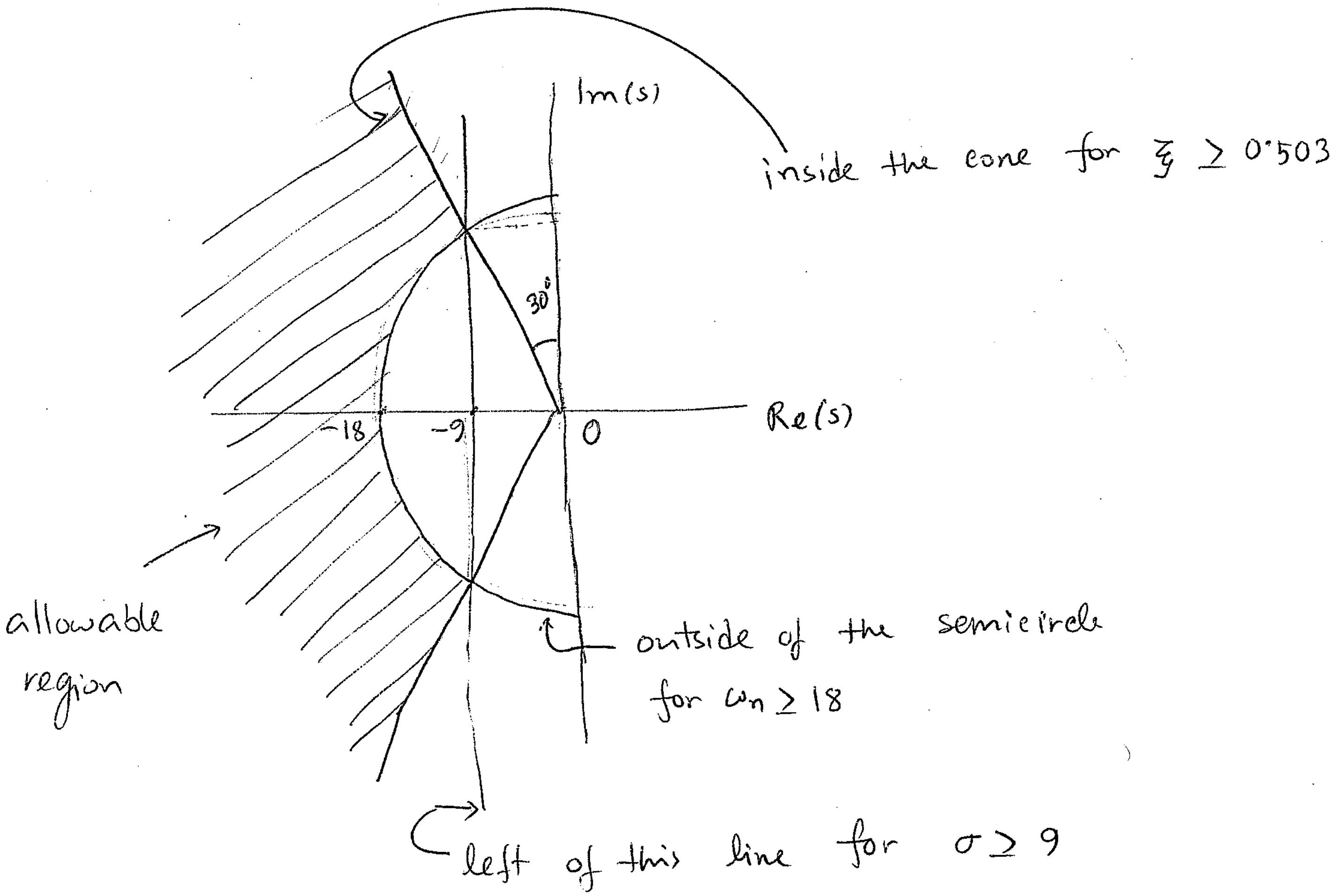
need $t_r = \frac{1.8}{\omega_n} \leq 0.1 \Rightarrow \omega_n \geq 18$.

$t_s = \frac{4.6}{\sigma} \leq 0.511 \Rightarrow \sigma \geq 9$.

$M_p = e^{-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}} \leq 0.16 \Rightarrow \zeta \geq \sqrt{\frac{(\ln .16)^2}{\pi^2 + (\ln .16)^2}} = 0.503$

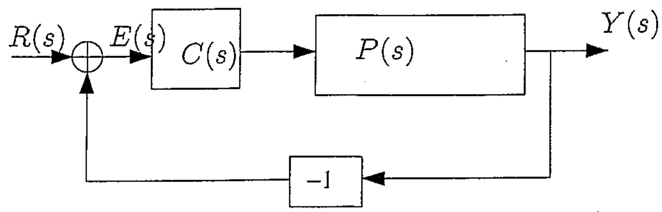
$\zeta = \sin\theta \Rightarrow \theta \geq 30.2^\circ$

(approximately 30°)



Problem 6. [20 pt]

A proportional controller is to be designed (for the feedback loop shown in the figure) to meet the following transient performance specifications: (i) rise time $t_r \le 0.2$ sec, (ii) settling time $t_s \le 10$ sec, and (iii) peak overshoot $M_p \le 0.1$ in response to a unit step command in the reference $r(t)$. The plant is $P(s) = \dfrac{1}{s^2 + .2s + 7}$.



1. Determine the expressions for the damping ratio and natural frequency of the closed loop transfer function from $R(s)$ to $Y(s)$ as a function of the proportional gain k_p.

2. Among the three parameters: rise time, settling time, and peak overshoot of the closed loop, which can be changed by varying k_p?

3. Determine the minimum value of the proportional gain k_p so that the closed loop rise time specification is met.

4. For this value of k_p, what are the values of the peak overshoot and the settling time of the closed loop?

$$H = \frac{Y}{R} = \frac{PC}{1+PC} = \frac{\frac{1}{s^2+0.2s+7}\cdot k_p}{1+\frac{1}{s^2+0.2s+7}\cdot k_p} \qquad \left\{\begin{array}{l}\text{proportional controller :}\\ C(s) = k_p\end{array}\right\}$$

$$= \frac{k_p}{s^2 + 0.2s + (7+k_p)}.$$

1 natural frequency $\omega_n = \sqrt{7+k_p}$

damping ratio $\zeta = \dfrac{.2}{2\omega_n} = \dfrac{1}{10\omega_n} = \dfrac{1}{10\sqrt{7+k_p}}$ Ans.

2 rise time $t_r = \dfrac{1.8}{\omega_n} = \dfrac{1.8}{\sqrt{7+k_p}}$ (H can be made to have stable complex poles by choosing $k_p > -6.99$, so these formulas are applicable)

$t_s = \dfrac{4.6}{\sigma} = \dfrac{4.6}{0.1}$, $M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$; and ζ is a function of k_p.

So ~~tn~~ t_r and M_p can be changed by varying k_p

but t_s cannot be.

3. Need $t_r = \dfrac{1.8}{\sqrt{7+k_p}} < 0.2 \Rightarrow \sqrt{7+k_p} \geq 9$

$\Rightarrow k_p \geq \underline{\underline{74}}$ minimum value

4. For $k_p = 74$, $\zeta = \dfrac{1}{10.9} = \dfrac{1}{90}$

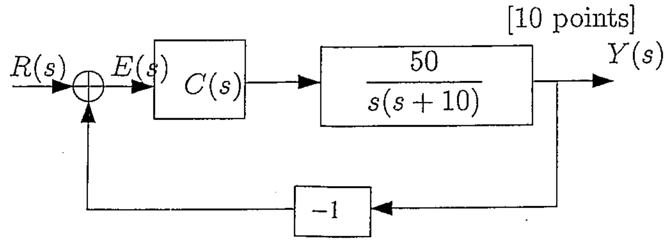
$\omega_n = 0.2$

$\Rightarrow M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.9657$

and $t_s = \dfrac{4.6}{\sigma} = \dfrac{4.6}{.1} = 46$ Ans

Problem 7. [10 points]

Consider the closed loop system shown in the figure, where $C(s) = \frac{s+z}{s+2}$. Plot the (i) **real axis part** and (ii) the **asymptotes** of the root locus of the closed loop with respect to the parameter z.



$$H = \frac{Y}{R} = \frac{PC}{1+PC} = \frac{\frac{s+z}{s+2}\cdot\frac{50}{s(s+10)}}{1+\frac{s+z}{s+2}\cdot\frac{50}{s(s+10)}} = \frac{(s+z)50}{(s+2)(s^2+10s)+(s+z)50}$$

$$\Rightarrow H = \frac{50(s+z)}{s^3+12s^2+20s+50s+z\cdot 50} = \frac{\frac{50(s+z)}{s^3+12s^2+70s}}{1+z\cdot\frac{50}{s^3+12s^2+70s}}$$

$\therefore$ characteristic eqn is $1 + z\cdot\frac{50}{s(s^2+12s+70)} = 0$.

$k = z$,

$L(s) = \frac{50}{s(s^2+12s+70)}$.

poles of $L = 0, -6 \pm 5.83j$

relative degree $= 3$

real axis part of root locus

the entire negative real axis

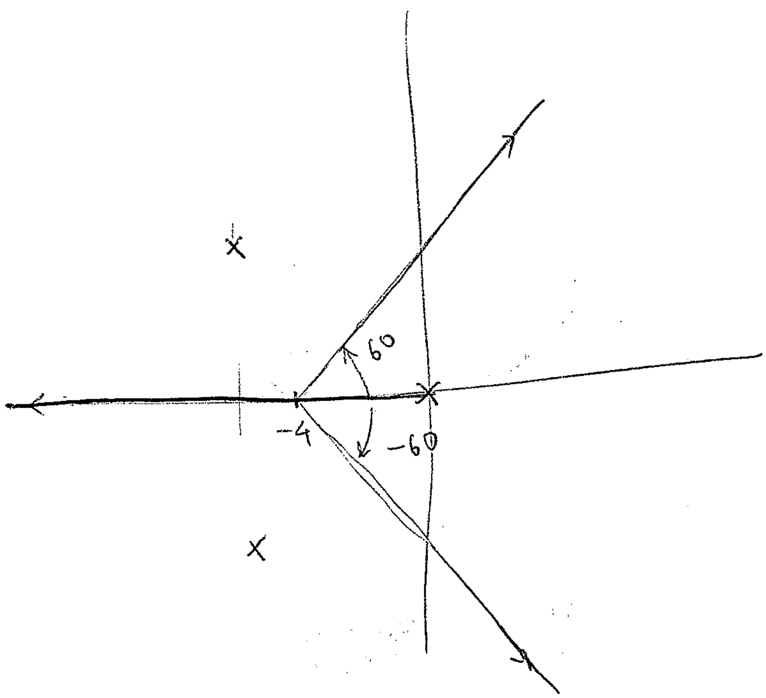
(odd # of poles)

asymptotes:

$\alpha = \frac{-12}{3} = -4$

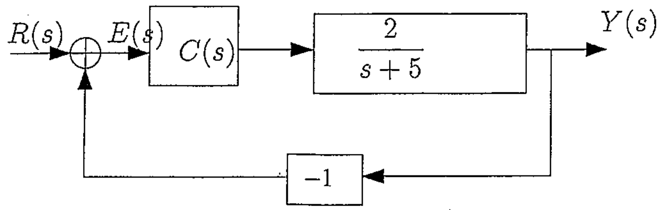
$\phi_l = \frac{180}{3}, \frac{180}{3}+\frac{360}{3}, \frac{180}{3}+\frac{360}{3}\cdot 2$

$= 60, 180, 300$



Problem 8. [10 points]

Consider the closed loop control system shown in the figure (similar to the cruise control system in the mini project), where $C(s) = 1 + \frac{k_i}{s}$ is a PI controller. Sketch the root locus of the closed loop poles with respect to the integral gain k_i.



$$H = \frac{Y}{R} = \frac{PC}{1+PC} = \frac{\frac{s+k_i}{s} \cdot \frac{2}{s+5}}{1 + \frac{s+k_i}{s} \cdot \frac{2}{s+5}} = \frac{2(s+k_i)}{s^2 + 5s + 2s + 2k_i}$$

$$\Rightarrow H = \frac{2 \frac{s+k_i}{s(s+7)}}{1 + k_i \frac{2}{s(s+7)}}$$

characteristic eqn: $1 + k_i \underbrace{\frac{2}{s(s+7)}}_{L(s)} = 0$

$L(s) = \frac{2}{s(s+7)}$

zeros: none

poles: 0 and −7

relative degree = 2

$$\Rightarrow \alpha = \frac{-7-0}{2} = -3'5$$

$$\phi_\ell = \frac{180^\circ}{2} + \frac{360}{2}(\ell-1), \quad \ell = 1,2$$

$$\Rightarrow \phi_{1,2} = 90^\circ, 270^\circ$$