Notes on Laplace transform, transfer function, and BIBO stability

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1 Laplace transform

The Laplace transform of a signal $\{y(t)\}$ is defined by

$$Y(s) = \mathcal{L}\left(y(t)\right) \triangleq \int_{0^-}^{\infty} y(\tau)e^{-s\tau}d\tau, \quad s \in \mathbb{C} \tag{1}$$

The integral exists (has a well defined, finite value) only if the signal $y(t)$ grows with t at a rate slower than the exponential e^{-st} decays, or vice versa. The rate of decay depends on the complex number s. Therefore, for a given signal $y(t)$, for the Laplace transform to exist, the value of s must be such that the integral above converges. The *region of convergence* of a Laplace transform $Y(s)$ is the region in the complex plane $\mathbb{C}$ that s can take values in so that $Y(s)$ is finite and well defined.

Example 1. Let us evaluate the Laplace transform of the complex signal

$$x(t) = e^{(a+bj)t} \triangleq \begin{cases} e^{a+bj}t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

By applying the definition of the Laplace transform, we get

$$\begin{aligned} \mathcal{L}(y(t)) = \mathcal{L}(x(t)) = X(s) = \int_0^{\infty} x(t)e^{-st}dt &= \int_0^{\infty} e^{(a+bj-s)t}dt \\ &= \frac{1}{a+bj-s}\left(\lim_{\tau\to+\infty} e^{(a-Re(s)+(b-Im(s))j)\tau} - 1\right) \\ &= \begin{cases} \frac{1}{s-(a+bj)} & \text{if } Re(s) > a \\ \text{undefined} & \text{otherwise} \end{cases} \end{aligned}$$

Hence, the Laplace transform of the signal $e^{(a+bj)t}$ is $\frac{1}{s-(a+bj)}$ with a region of convergence $Re(s) > a$. In general, the Laplace transform of a signal e^{pt} (where p is a complex number) is $\frac{1}{s-p}$ with a ROC given by $Re(s) > Re(p)$. Note that the strict inequality is important.

The Laplace transform $Y(s)$ of a signal $y(t)$ is a complex number, whose values depend on the complex argument s. Therefore Laplace transform can be thought of as a complex function, that maps one complex plane (the s-plane) to another complex plane (the $Y(s)$-plane). For a complex number $s_0 \in$ ROC, the Laplace transform $Y(s_0)$ is a complex number.

Exercise: plot the values of $Y(s)$ for a few complex numbers s.

1.1 Useful properties of the Laplace transform

The following properties of the Laplace transform will be very useful in analyzing LTI systems specified in terms of differential equations. To state one of the properties, we'll need the *unit step* signal $1(t)$, which is defined as:

$$1(t) \triangleq \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{2}$$

1. *Linearity*:

$$\mathcal{L}(ax(t) + by(t)) = a\mathcal{L}(x(t)) + b\mathcal{L}(y(t)),$$

   where $\{x(t)\}, \{y(t)\}$ are signals are a, b are complex scalars.

2. *Transform of a derivative signal:* If $Y(s)$ is the Laplace transform of a signal $\{y(t)\}$, then the transform of the signal $\dot{y}$ is

$$\mathcal{L}\left(\dot{y}(t)\right) = sY(s) - y(0).$$

   This can be shown by integration by parts:

$$\begin{aligned} \mathcal{L}\left(\dot{y}\right) &= \int_0^\infty e^{-st}\dot{y}(t)dt \\ &= \left[e^{-st}y(t) - \int(-s)e^{-st}y(t)dt\right]_0^\infty = -y(0) + sY(s), \end{aligned}$$

   where we have used the fact that, since $e^{-st}y(t)$ is integrable for all values of s in the ROC (otherwise the Laplace transform $Y(s)$ would not exist[1]), $e^{-st}y(t) \to 0$ as $t \to \infty$.

3. *Transform of a integral signal:* If $Y(s)$ is the Laplace transform of a signal $\{y(t)\}$, then

$$\mathcal{L}\left(\int_0^t y(\eta)d\eta\right) = \frac{1}{s}Y(s)$$

   It is left as an exercise for you to show that this true, using the previous result.

4. *Time shifting*: If $Y(s)$ is the Laplace transform of $y(t)$, then the Laplace transform of the *shifted signal* $y(t-\tau)1(t-\tau)$ is

$$\mathcal{L}\left(y(t-\tau)1(t-\tau)\right) = e^{-s\tau}Y(s).$$

   To prove it,

$$\begin{aligned} y(t-\tau)1(t-\tau) \xrightarrow{\mathcal{L}} \int_\tau^\infty y(t-\tau)e^{-st}dt &= e^{-s\tau}\int_\tau^\infty y(t-\tau)e^{-s(t-tau)}dt \\ &= e^{-s\tau}\int_0^\infty y(\eta)e^{-s\eta}d\eta \quad \text{where } \eta \triangleq t-\tau \\ &= e^{-s\tau}Y(s). \end{aligned}$$

   Note that $\mathcal{L}(y(t-\tau)) \neq e^{-s\tau}Y(s)$!

   When the time shift τ is positive, $y(t-\tau)$ is the signal $y(t)$ *delayed* by τ units of time. When τ is negative, $y(t-\tau)$ is the signal $y(t)$ *advanced* by τ units of time.

Sketch the signals $e^{-t}1(t)$, sketch the signal that is obtained by delaying it by 1 sec, and the signal obtained by advancing it by 1 sec. Find mathematical expressions for the delayed as well as the advanced versions.

[1] why?

5. *Convolution:* The convolution $x(t) * y(t)$ of two signals $x(t)$ and $y(t)$ is defined as

$$x(t) * y(t) \triangleq \int_0^t x(\tau)y(t-\tau)d\tau. \tag{3}$$

The Laplace transform of the convolution of two signals is given by the product of their Laplace transforms:

$$z(t) = x(t) * y(t) \qquad \Leftrightarrow \qquad Z(s) = X(s)Y(s). \tag{4}$$

To prove it,

$$\begin{aligned}\mathcal{L}(x(t) * y(t)) &= \int_0^\infty (x(t) * y(t))\, e^{-st}dt \\ &= \int_0^\infty \left(\int_0^t x(\tau)y(t-\tau)d\tau\right) e^{-st}dt \\ &= \int_0^\infty \int_0^t x(\tau)y(t-\tau)e^{-st}d\tau dt\end{aligned}$$

We will now reverse the order of the integration. To do so, notice that in the $t-\tau$ plane (t along the horizontal axis and τ along the vertical axis), the area over which the integrand is integrated consists of that part of the first quadrant that lies below the line $t = \tau$. The integration above is done by first letting τ vary from 0 to t and then letting t vary from 0 to ∞. To reverse the order of integration, that is, to integrate first over t and then over τ, while integrating over the same area, we must first let t vary from τ to ∞ and then let τ vary from 0 to t. This implies that

$$\begin{aligned}\mathcal{L}(x(t) * y(t)) &= \int_0^t \int_\tau^\infty x(\tau)y(t-\tau)e^{-st}dtd\tau \\ &= \int_0^t x(\tau)\left(\int_\tau^\infty y(t-\tau)e^{-st}dt\right)d\tau \\ &= \int_0^t x(\tau)\left(\int_0^\infty y(t-\tau)1(t-\tau)e^{-st}dt\right)d\tau \\ &= \int_0^t x(\tau)e^{-s\tau}Y(s)d\tau = \left(\int_0^t x(\tau)e^{-s\tau}d\tau\right)Y(s) = X(s)Y(s),\end{aligned}$$

where we have used the previously proved result that $\mathcal{L}(y(t-\tau)1(t-\tau))$ is $e^{-s\tau}Y(s)$.

1.2 Useful pairs of signals and their Laplace transforms

The *impulse* or the *Dirac delta* function $\delta(t)$ is defined as:

$$\int_{-\infty}^\infty f(t)\delta(t)dt = \int_{0^-}^{0^+} f(t)\delta(t) = f(0) \tag{5}$$

for every function $f(t)$. It can be shown that this implies $\int_{0^-}^{0^+} \delta(t)dt = 1$[2]. The impulse function is 0 everywhere except at 0, where it is arbitrarily large, but in such a manner that the total area under it is 1.

The unit step signal has been defined earlier.

[2] How?

The *unit ramp* signal $t(t)$ is defined as:

$$t(t) \triangleq \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{6}$$

It is left as an exercise for you to show the following:

1. $\delta(t) \xrightarrow{\mathcal{L}} 1$.
2. $1(t) \xrightarrow{\mathcal{L}} \frac{1}{s}$.
3. $t(t) \xrightarrow{\mathcal{L}} \frac{1}{s^2}$.
4. $\sin(\omega_0 t) \xrightarrow{\mathcal{L}} \frac{\omega_0}{s^2 + \omega_0^2}$.
5. $\sin(\omega_0 t) \xrightarrow{\mathcal{L}} \frac{s}{s^2 + \omega_0^2}$.

The signals mentioned above are the most important ones in the analysis of LTI systems and in the design of linear control systems.

1.3 Inverse Laplace transform:

The inverse of a Laplace transform $Y(s)$ defined as

$$y(t) = \mathcal{L}^{-1}(Y(s)) \triangleq \int_{\sigma_c - j\infty}^{\sigma_c + j\infty} Y(s) e^{st} ds, \tag{7}$$

where σ_c is any real number so that $\sigma_c + j\omega$ is inside the ROC of $Y(s)$ for every possible ω. Looking at the limits of integration, it is obvious that the integration is being carried out in the complex plane along a "vertical" line passing through the real axis at σ_c. You can think of the inverse transform as something that recovers the signal $y(t)$ from the Laplace transform $Y(s)$.

The inverse transform is also linear. That is, if $X(s)$ and $Y(s)$ are Laplace transforms of two signals $x(t)$ and $y(t)$ with associated ROCs A and B such that $A \cap B$ is not empty, then the inverse Laplace transform of $X(s) + Y(s)$ is given by $\mathcal{L}^{-1}(X(s) + Y(s)) = x(t) + y(t)$.

Evaluation of the inverse transform involves complex integration, but usually the signal can be recovered from $Y(s)$ by partial fraction expansion and using knowledge of Laplace transform of complex exponential signals.

2 Transfer function

Consider the following linear differential equation that models the dynamics of a mass m that is acted on by an external force u, when the mass has a spring and damper attached to it:

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = u(t), \qquad \text{I. C. :} x(0), \dot{x}(0). \tag{8}$$

What does this mean to take Laplace transform of both sides?

Take the Laplace transform of both sides to get

$$m(s^2X(s) - sx(0) - \dot{x}(0)) + b(sX(s) - x(0)) + kX(s) = U(s)$$
$$\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right) X(s) = U(s) + \frac{s+b}{m}x(0) + \dot{x}(0)$$

which can be rearranged into

$$Y(s) = \underbrace{\frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}}}_{H_c(s)} U(s) + \underbrace{\frac{\frac{s+b}{m}x(0) + \dot{x}(0)}{s^2 + \frac{b}{m}s + \frac{k}{m}}}_{Y_{I.C.}(s)} \tag{9}$$

$$Y(s) = \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} U(s) + \frac{\frac{x(0)}{m}s + (\frac{b}{m}x(0) + \dot{x}(0))}{s^2 + \frac{b}{m}s + \frac{k}{m}}, \tag{10}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left(H_c(s)U(s)\right) + \mathcal{L}^{-1}\left(Y_{I.C.}(s)\right). \tag{11}$$

This shows that the output consists of two terms, one due to the input and the other due to the initial conditions.

We will deal extensively with causal linear time invariant (LTI) systems with a single input and a single output (SISO) described by a (linear, constant coefficient) ordinary differential equation of the following form[3]:

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-2}\ddot{y} + a_{n-1}\dot{y} + a_n y = b_1 u^{(m)} + b_2 u^{(m-1)} + \cdots + b_{m-1}\ddot{u} + b_m \dot{u} + b_{m+1} u \tag{12}$$

where $y^{(n)}$ is the n-th derivative of y, and a_i, b_j's are parameters that *do not change with time*. Taking Laplace transform of both sides of the equation above, we get

$$Y(s) = H_c(s)U(s) + Y_{I.C.}(s), \tag{13}$$

where the complex function

$$H_c(s) \triangleq \frac{\alpha_m s^m + \alpha_{m-1}s^{m-1} + \cdots + \alpha_1 s + \alpha_0}{s^n + \beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0} \tag{14}$$

is called the *(continuous time) transfer function* from u to y for the system (12), and $I_c(s)$ depends (as it does in (9)) on the initial conditions.

Upon taking the inverse Laplace-transform of (13), we get

$$y(t) = \mathcal{L}^{-1}\left(H_c(s)U(s)\right) + \mathcal{L}^{-1}\left(Y_{I.C.}(s)\right)$$

which shows that the output of the system is a superposition of the response due to the input and the response due to initial conditions.

If all the initial conditions are 0, then (13) reduces to

$$Y(s) = H_c(s)U(s) \tag{15}$$

Since $y(t)$ can be recovered by taking the inverse Laplace transform of the right hand side, it shows that when all initial conditions are zero, the transfer function together with the input completely determines the output.

Poles and zeros The roots of the denominator polynomial of a transfer function $H_c(s)$ are called the *poles* of the system and the roots of its numerator polynomial are called the *zeros* of the system.

note that the transfer function is from $U(s)$ to $Y(s)$, not from $u(t)$ to $y(t)$. SO *continuous-time* transfer function is really a misnomer.

In MATLAB, `tf(num, den)` creates a continuous-time transfer function with numerator and denominator specified by `num, den`. Example: `tf(1, [1 2 3])` produces $\frac{1}{s^2+2s+3}$.

In MATLAB, `zpk(z, p, k)` creates a continuous-time transfer function with zeros, poles and gain specified by `z`, `p` and `k`.

[3] Note that the coefficient of the highest derivative of y is 1. This is the standard way of expressing such equations.

2.1 Impulse response and transfer function

So far we have been talking about Laplace transform of signals. Now we have a Laplace transform $H_c(s)$ that we obtained by manipulating a differential equation. A question might arise at this point: what is the signal whose Laplace transform is $H_c(s)$?. The answer to this question is: *the impulse response*. The impulse response, usually denoted by $h(t)$, of a LTI system is its output when it is driven by an impulse input $\delta(t)$ with all initial conditions set to 0. It is impossible to determine the impulse response experimentally, though it is a useful concept for analysis.

To verify the above claim, let us do the following thought experiment on a system whose transfer function from the input u to the output y is $H(s)$. Let the system now be driven by an impulse input $\delta(t)$. In that case,

$$Y(s) = H(s)\mathcal{L}(\delta(t)) = H(s),$$

The MATLAB command `impulse(sys)` can be used to plot the impulse response for the system specified in `sys`.

since the Laplace transform of $\delta(t)$ is 1. The impulse response $h(t)$ is the inverse Laplace transform of $Y(s)$, so that we get

$$h(t) = \mathcal{L}^{-1}\left(H(s)\right) \Leftrightarrow H(s) = \mathcal{L}\left(h(t)\right)$$

Lesson: The transfer function of a system (from the input to the output) is therefore the Laplace transform of its impulse response.

For an arbitrary input $u(t)$, since the output $y(t)$ satisfies $Y(s) = H(s)U(s)$ when initial conditions are zero, from the convolution property of Laplace transforms, we get

$$y(t) = h(t) * u(t).$$

Thus, the response to an arbitrary input can be determined from the knowledge of the impulse response of the system (when initial conditions are 0).

3 Stability

A system is called *BIBO stable* if the output is bounded for every bounded input. It has two equivalent characterizations for LTI systems specified by transfer functions, in terms of the impulse response, and in terms of its poles.

1. A LTI system is BIBO stable if and only if its impulse response $h(t)$ is absolutely integrable, i.e.,

$$\text{BIBO stable } \Leftrightarrow \int_0^\infty |h(t)|dt < \infty.$$

Can you find a bounded input such that if $h(t)$ is not absolutely integrable, the response to this input will be unbounded?

2. A LTI system is BIBO stable if and only if the real part of each pole of its transfer function $H_c(s)$ is negative.

If a system is BIBO stable, it can be shown that

$$\epsilon(t) \triangleq \mathcal{L}^{-1}\left(Y_{I.C.}(s)\right) \tag{16}$$

is a signal such that $\epsilon(t) \to 0$ as $t \to \infty$. Therefore, for a BIBO stable system, the output at steady state (as $t \to \infty$) is given by

$$y_{ss}(t) = \mathcal{L}^{-1}\left(H_c(s)U(s)\right) \tag{17}$$

Thus the transfer function provides information on steady state behavior of the system.