Optimal Estimation University of Florida
Mechanical and Aerospace Engineering

Scientific questions often have a surface appearance of dumbness .... They are asked in order to prevent dumb mistakes later on.
- Robert Pirsig, in *Zen and the art of motorcycle maintenance*

HW 2

Issued: September 2, 2009, Due: September 9, 2009 (in class)

Problem 1. In this homework (and all subsequent ones), correct answers must be accompanied by adequate explanation for points to be awarded. Is it the same as saying "correctness is a necessary condition for earning points but not sufficient"? Read the article

`http://www.themathpage.com/abooki/logic.htm`

to learn about valid arguments, necessary conditions, and sufficient conditions.

Problem 2. Is the condition "$y \in \mathcal{R}(A)$" necessary for a vector x to exist that satisfies $Ax = y$? Or is it sufficient? Or both? Given an $m \times n$ matrix A and a $m \times 1$ vector y, how would you numerically check whether $y \in \mathcal{R}(A)$ or not?

Problem 3. We know that a for the solution to the equation $Ax = y$ to be unique, we must have $\mathcal{N} = \{0\}$. Given an $m \times n$ matrix A, how would you numerically check whether $\mathcal{N}(A) = \{0\}$ or not?

Problem 4. Write a MATLAB program for simulating the parameter estimation experiment described in class with noisy inputs and outputs. Choose $a = -0.2$, $b = 10$, $T = 0.001$ second, and use a sinusoidal input. See the course website for an example.

1. Estimate the two parameters a_d and b_d using the least squares technique described in the first two lectures of the course.

2. Repeat the estimation exercise N times (say, $N = 500$) and collect the N estimates. Make sure that you use distinct sequences of random numbers to simulate noise in each of these experiments.

3. Plot a histogram of the least squares estimate of $\hat{a}_d$ data. (Please submit your MATLAB script, but make sure it is not longer than one page)

4. Vary the initial condition used in the simulations and look at the resulting estimates. Does the initial condition have an effect on the accuracy of the estimates obtained? If so, why, and if not, why not?

Combinatorics

Basic counting principle: if you can do one thing in r ways and another in s ways then you can both of them in $r \times s$ ways.

Problem 5. 1. How many different 7-place license plates are possible if the first 2-places are for letters and the last 5 places are for numbers?

2. Repeat part (1) under the assumption that no number or letter can be repeated in a license plate.

3. How many plates are possible if there is no restriction on where the numbers and letters appear? (the number of letters and numbers is still held fixed at 2 and 5, respectively.)

Problem 6. A committee of 3 men and 3 women is to be formed from a group of 8 men and 6 women. How many different committees are possible if

1. 2 of the men refuse to serve together?

2. a man and a woman refuse to serve together ?

Problem 7. The lock on the door of a computer center has five buttons numbered one through five. The combination of buttons that opens the lock is a sequence of five numbers that is reset every week.

1. How many combinations are possible if every button must be used once?

2. Assume that the lock can also have combinations that require you to push two buttons simultaneously and the other three one at a time. How many more combinations does this permit?

Probability questions

Problem 8. Prove, by using the axioms of probability that

1. $P(\phi) = 0$.

2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Problem 9. Two die are thrown.

1. Describe the sample space Ω.

2. What are the favorable outcomes for the event that the two numbers that turn up add up to 8? What is the probability of this event?

3. What is the total number of events (i.e., the number of distinct subsets of Ω, or, the number of elements of the σ-field F)?

Problem 10. What is the probability that in a birthday party attended by N guests (excluding the host, that is, the "birthday boy/girl"), at least[1] one guest has the same birthday as the host? (State clearly any assumption you make in calculating the probability.) If you are willing to bet that at least one guest in the party has the same birthday as the host only if the probability of you winning is greater than 0.5, what should be the minimum value of N for you to bet?

[1] In problems involving "at least", it is often convenient to use the fact that the probability of "at least 1 success in n Bernoulli trials is equal to one minus the probability of 0 successes in n trials."

Problem 11. What is the probability that in a party attended by N people, at least two people have the same birthday? How many people should there be in the party before you are willing to bet that at least two people have the same birthday (if you play by the criterion of the previous problem)?

Problem 12. Total probability Prove that

1. if $A_1 \cup A_2 = \Omega$, A_1 and A_2 are mutually exclusive, $P(A_1) > 0$, $P(A_2) > 0$ and B is an event defined in the same probability space that A_1 and A_2 are defined, then $P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2)$.
2. (general case) if $A_i, i = 1, \ldots, n$ are such that each distinct pair of A_i's are mutually exclusive and $\cup_{i=1}^{n} A_i = \Omega$ with $P(A_i) > 0 \ \forall i$, then for every event B defined over the probability space where A_i's are defined,

$$P(B) = P(B|A_i)P(A_i) + P(B|A_2)P(A_2) + \cdots + P(B|A_n)P(A_n)$$

   (This is known as *Baye's Theorem.*)

Problem 13. Prove that if A and B are two events and $P(B) > 0$, then

$$P(A|B) = 1 - P(A^c|B)$$

(This result was used in the problem that was solved in class)

Problem 14. In answering a question on a multiple-choice test, a test-taker either knows the answer or guesses. Let p be the probability that a test-taker knows the answer and $1 - p$ be the probability that the he guesses. Assume that when guessing, the probability of getting the right answer is $1/m$, where m is the number of multiple-choice alternatives. What is the conditional probability that the test-taker knew the answer to a question, given that he answered it correctly? Compute it for $m = 4$ and $p = 0.6$.

Problem 15. Two students who are working on the problem above are arguing about what the underlying "random experiment" is. According to student A, the random experiment is such that it has the following two possible outcomes: (i) the test-taker knows the answer, and (ii) the test-taker guesses. According to student B, the experiment is one whose possible outcomes are the following: (i) the test-taker's answer is correct, and (ii) the test-taker's answer is incorrect. As an expert in probability, you are asked to weigh in and resolve the issue. Describe what you would say to the poor confused students.

Questions on random variables

Problem 16. Consider the function

$$g(x) = \begin{cases} \frac{x}{c} e^{-\frac{x^2}{2\sigma^2}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

find the value of c (as a function of σ^2) for g to qualify as a probability density function.

Problem 17. The amount of time, in hours, that a computer functions before breaking down is a continuous random variable X whose p.d.f. is given by

$$f_X(x) = \begin{cases} \lambda e^{-x/50} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

1. What is the value of λ?
2. What is the probability that a computer will function between 100 and 200 hours before breaking down?
3. What is the probability that it will function less than 100 hours?