Solution to HW 2
EML 6934, Fall '09

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Problem 1 Statement 1: "correct answers must be accompanied by adequate explanations for points to be awarded."

This statement can be reworded as

$$\sim E \Rightarrow \sim P$$

where "$\sim$" means 'not' and E : accompanied by adequate explanations

P : points are awarded.

(since all that the statement is saying is that if you donot provide adequate explanation, points will not be awarded. The statement does not preclude the possibility of a crazy instructor ~~&~~ who may fail to award points even if adequate explanation is provided.)

Statement 2 "correctness is a necessary condition but not sufficient for earning points."

This can be written as $\sim C \Rightarrow \sim P$; $C \not\Rightarrow P$

where C : answer is correct

P : (as above)

clearly the two statements are not the same; one does not even talk about correctness (C) while the other does not include anything about explanation (E).

Problem 2 : S1 : $y \in R(A)$

S2: $\exists\, x$ s.t. $Ax = y$

S1 is a necessary and sufficient condition for S2. First, if S1 is true, then by definition of $R(A)$, there exists a vector x s.t. $Ax = y$. This implies S1 is sufficient for S2. if S1 is not true, ~~theren~~ then again, by definition of $R(A)$ it follows that S2 is not true. This shows S1 is necessary for S2 as well.

To numerically check whether $y \in R(A)$, note that if $y \in R(A)$ then $\exists\, x$ s.t. $Ax = y$. $\Leftrightarrow$ y is a linear combination of the columns of A $\Leftrightarrow$ If $a_1, a_2, \cdots a_n$ are the columns of A, y is linearly dependent on the set of vectors $\{a_1, a_2, \cdots, a_n\}$ $\Leftrightarrow$ the number of linearly independent vectors in the two sets $\{a_1, a_2, \cdots, a_n\}$ and $\{a_1, a_2, \cdots a_n, y\}$ are the same $\Leftrightarrow$ ~~column~~ number of linearly independent columns in the matrix A is the same as that in $[A|y]$ where $[A|y] = [a_1, a_2, \cdots a_n, y]$. $\Leftrightarrow$ $\operatorname{rank}[A|y] = \operatorname{rank}[A]$

Therefore, computing the ranks of A and $[A|y]$ and checking whether they are the same will tell you whether $y \in R(A)$ or not.

Problem 3 : A similar argument shows that $N(A) = \{0\}$ if and only if $\operatorname{rank}[A] = \#$ of columns of A, which can be used to check if $N(A) = \{0\}$ or not.

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5.1 $\underset{L}{\_}\ \underset{L}{\_}\ \underset{N}{\_}\ \underset{N}{\_}\ \underset{N}{\_}\ \underset{N}{\_}\ \underset{N}{\_}$

2 letters can be chosen in 26×26 ways (each place can take any letter, since repetitions are allowed)

5 numbers can be chosen in $10 \times 10 \times 10 \times 10 \times 10 = 10^5$ ways

$\Rightarrow$ Number of plates $= 26^2 \times 10^5 = 676 \times 10^5$

5.2 1st letter can be chosen in 26 ways while the second letter can only be chosen in 25 ways, since no repetitions are allowed. Arguing similarly for the numbers, the answer is $26 \times 25 \times 10 \times 9 \times 8 \times 7 \times 6$

$= 19,656,000$

5.3 since repetitions are allowed, ~~and rearrangement~~ choose 2 places among the 7 for the letters, which can be done in $\binom{7}{2}$ ways. Once this is done, the locations for the numbers become fixed as well. Now each place for a letter can be filled in 26 ways, and each place for a number can be filled in 10 ways.

So the total # of distinct licence plates is

$$\binom{7}{2} \cdot 26^2 \cdot 10^5 = \cdots$$

$\boxed{6.1}$ Without any restriction, the number of committees is $\binom{8}{3}\binom{6}{3}$, since 3 Men out of 8, and 3 women out of 6 are to chosen and the order does not matter.

The number of committees in which the two men who refuse to serve together <u>do appear</u> is $\binom{6}{1}\binom{6}{3}$, since only one out of the remaining six need to be chosen.

$\Rightarrow$ The number of committees in which these two men do not appear is $\left[\binom{8}{3} - \binom{6}{1}\right]\binom{6}{3} = 1000.$

6.2 Again, w/o restriction, $\binom{8}{3} \times \binom{6}{3}$ are possible.

The number of committees in which the man and the woman who refuse to serve together <u>do appear</u> is $\binom{7}{2} \times \binom{5}{2}$, since now you have to choose 2 men out of the remaining 7 and 2 women out of the remaining 5.

$\Rightarrow$ The number is $\binom{8}{5}\binom{6}{3} - \binom{7}{2}\binom{5}{2} = 910.$

$\boxed{7.1}$ $5 \times 4 \times 3 \times 2 \times 1 = 5!$

1st button ↑ second button ($\because$ every button must be used once and we have equal number of buttons and numbers, there cannot be any repetition)

7.2 The two buttons that are to be pressed together must be distinct, since there is no such thing as pressing 1 and 1 together. Since order does not matter, These 2 buttons can be chosen ~~to~~ in $\binom{5}{2}$ ways (3)

$$\left.\begin{array}{cccc} 2B & N & N & N \\ N & 2B & N & N \\ N & N & 2B & N \\ N & N & N & 2B \end{array}\right\} \Rightarrow$$

the total number of possibilities with 2 buttons and 3 numbers is

$$4 \times \left(\binom{5}{2} \times \underbrace{5 \times 5 \times 5}_{} \right) = 4 \times 10 \times 5^3 = 5000.$$

since (12) 1 1 1 is allowed.

$\therefore$ the number of additional combinations $= 5000.$

$\boxed{8}$ **8.1** For an arbitrary event A, we have $A \cap \phi = \phi$

$\Rightarrow$ A and ϕ are mutually exclusive. And, $A \cup \phi = A$

Applying the third axiom of probability, $P(A \cup \phi) = P(A) + P(\phi)$ — (1)

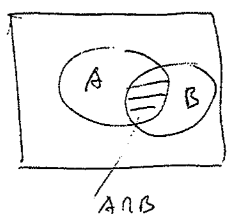
but since $A \cup \phi = A$

$P(A \cup \phi) = P(A)$ — (2)

subtracting (1) from (2), we get $P(\phi) = 0$

8.2

$$A \cup B = \left[A - (A \cap B) \right] \cup B$$



Caution! this "$-$" sign is not an arithmatic subtraction. Define $C \triangleq A - A \cap B$

$$C = \{ \omega \mid \omega \in A \text{ but } \omega \notin A \cap B \}.$$

by construction, C and B are mutually exclusive.

$\Rightarrow$ $P(A \cup B) = P(C \cup B) = P(C) + P(B)$ (Axiom 3)

To evaluate $P(C)$, note that $A = C \cup (A\cap B)$ and,

C and $(A\cap B)$ are mutually exclusive,

$$\Rightarrow P(A) = P(C) + P(A\cap B)$$

$$\Rightarrow P(C) = P(A) - P(A\cap B)$$

Combining this with the previous equation, we get

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

9.1 $\Omega = \{ 1|1, 1|2, 1|3, \ldots, 1|6, 2|1, 2|2, \ldots 2|6, \ldots\ldots, 6|6 \}$

total $6 \times 6 = 36$ outcomes;

9.2 the favorable outcomes for the event **E** that the two numbers add upto 8 are 2|6, 6|2, 3|5, 5|3, 4|4,

i.e., $E = \{ 2|6, 6|2, 3|5, 5|3, 4|4 \}$

$$P(E) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}} = \frac{5}{36}$$

Here we are using the interpretation of probability as the ratio of number of favorable outcomes to the total number of outcomes.

9.3 total # of events = total # distinct subsets of Ω

= # subsets of Ω with 1 element +

" " 2 elements +

" " 3 "

+ - - - 35 elements

$$+ \underset{\text{from } \phi}{\underset{\uparrow}{1}} + \underset{\text{from } \Omega}{\underset{\uparrow}{1}}$$

$$= \binom{36}{1} + \binom{36}{2} + \cdots + \binom{36}{34} + \binom{36}{35} + 1 + 1$$

$$= \binom{36}{0} + \binom{36}{1} + \cdots + \binom{36}{35} + \binom{36}{36}$$

$$= (1+1)^{36} = 2^{36}$$

$$\left[(p+q)^n = \binom{n}{0} p^n + \binom{n}{1} p^{n-1} q + \binom{n}{2} p^{n-2} q^2 + \cdots + \binom{n}{n-1} p\, q^{n-1} + \binom{n}{n} q^n \right.$$

put $p = q = 1$ $\left.\right]$

↑ this is called the Binomial theorem

$\boxed{10}$ p(at least one guest has the same birthday as the host)

$= 1 - p$(no guest has the same birthday as the host)

$= 1 - p(G_1)\, p(G_2) \cdots p(G_N)$

where G_i = guest # i does not have the same birthday as the host.

(I am assuming the events G_i are independent.)

For a particular i,

$$p(G_i) = \frac{364}{365} \begin{array}{l} \leftarrow \text{number of favorable outcomes} \\ \leftarrow \text{total \# of outcomes} \end{array}$$

the underlying random experiment being choosing the birthday of guest i and checking if it is the same as the given birthday of the host.

(Here I am assuming that every day is equally likely to be the birthday of guest i, for every i, so I can use this interpretation to compute $p(G_i)$)

$\Rightarrow$ p(at least one guest) - - - -) $= 1 - \left(\frac{364}{365}\right)^N$

To bet, we need $p(\cdots) \geq 0.5$

$$\Rightarrow \left(\frac{364}{365}\right)^N \leq 0.5$$

$$\Rightarrow N \geq \frac{\log(0.5)}{\log 364 - \log 365} \approx 252.65$$

(dividing by −ve #, inequality changes direction)

$\therefore$ Need at least 253 guests before betting.

11 Think of the experiment as picking birthdays of N people. Every outcome is a sequence of N numbers, each between 1 and 365. (ignore leap years !)

$$\omega = b_{g_1}\, b_{g_2} \cdots \underbrace{b_{g_i}}_{\text{birthday of guest } i} \cdots b_{g_N}$$

E = no two guests have the same birthday

= { ω | ω contains no repeated entry }

# of favorable outcomes for E = number of ways N distinct objects can be selected from 365 distinct objects, without regard to order = $365 \times 364 \times \cdots (365-N+1)$

total # of outcomes = 365^N

$$\Rightarrow P(E) = \frac{365 \cdot 364 \cdots\cdots (365-N+1)}{365^N}$$

$\Rightarrow$ P(at least one pair of guests have a common birthday)

$= 1 - P($ no two guests have the same birthday$)$

$$= 1 - \frac{365 \times 364 \times \cdots (365 - N + 1)}{365^N}$$

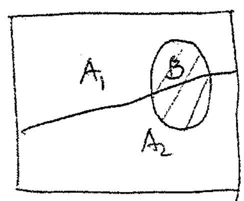
trying out a few values of N,

we get $(N = 22) \Longrightarrow 0.4757$

$(N = 23) \longrightarrow 0.5073$

$\therefore$ Need at least <u>23 guest</u> before bet becomes favorable.

$\boxed{12}$ <u>91</u> $A_1 \cup A_2 = \Omega$ and $A_1 \cap A_2 = \phi$



$$B = B \cap \Omega$$
$$= B \cap (A_1 \cup A_2)$$
$$= (B \cap A_1) \cup (B \cap A_2)$$ $\leftarrow$ distributive law of set intersection and union

$\because$ A_1 and A_2 are mutually exclusive $(B \cap A_1)$ and $(B \cap A_2)$ are also mutually exclusive.

You can check this: $(B \cap A_1) \cap (B \cap A_2) = B \cap B \cap A_1 \cap A_2$ ~~$= B \cap B \cap A_1 \cap A_2$~~

$$= B \cap (A_1 \cap A_2)$$
$$= B \cap \phi$$
$$= \phi$$

Now, applying the third axiom,

$$P(B) = P(B \cap A_1) + P(B \cap A_2)$$
$$= P(B|A_1)P(A_1) + P(B|A_2)P(A_2)$$ by Baye's Rule.

$\boxed{14}$ G = guesses K = knows C = answers correctly.

Given $p(K) = p$, $p(G) = 1-p$, and that $K \cup G = \Omega$, $K \cap G = \phi$

also, $p(C|G) = \frac{1}{m}$

We want to find $p(K|C)$

$$p(K|C) = \frac{p(K \cap C)}{p(C)} = \frac{p(C|K)\,p(K)}{p(C)}$$

$p(C) = p(C|K)\,p(K) + p(C|G)\,p(G)$ by using problem 9.1

Assuming the student is rational, $p(C|K) = 1$

$$\Rightarrow p(K|C) = \frac{1 \cdot p}{1 \cdot p + \frac{1}{m}\cdot(1-p)} = \frac{mp}{mp + 1 - p} = \frac{mp}{1 + (m-1)p}$$

For $m = 4$, $p = 0.6$, $p(K|C) = 0.857$

$\boxed{16}$ For $g(x)$ to qualify as a pdf, $g(x) \geq 0 \;\forall x$ (which is clearly satisfied for $c > 0$) and $\int_{-\infty}^{\infty} g(x)\,dx = 1$

$$\int_{-\infty}^{\infty} g(x)\,dx = \int_{-\infty}^{0} g(x)\,dx + \int_{0}^{\infty} g(x)\,dx = 0 + \frac{1}{c}\cdot\int_{+0}^{\infty} x\, e^{-\frac{x^2}{2\sigma^2}}\,dx$$

$$= \frac{\sigma^2}{c}\int_{0}^{\infty} e^{-y}\,dy \qquad y = \frac{x^2}{2\sigma^2} \Rightarrow dy = \frac{x}{\sigma^2}\,dx$$

$$= \frac{\sigma^2}{c}$$

$\Rightarrow c$ must be σ^2 for $g(x)$ to be a pdf.

$\boxed{15}$ Both students are incorrect. The sample space should be

{(guesses, correct), (guesses, incorrect), (knows, correct), (knows, incorrect)}

Since events such as 'correct' cannot be defined if the sample space is {guesses, knows}, and vice versa, both students are wrong.

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1. $\because \int_{-\infty}^{\infty} f_X(x)\,dx = 1$, we have $\lambda \int_0^{\infty} e^{-\frac{x}{50}}\,dx = 1$

$$\Rightarrow 50\lambda \int_0^{\infty} e^{-y}\,dy = 1 \qquad y = \frac{x}{50} \Rightarrow dx = 50\,dy$$

$$\Rightarrow \lambda = \underline{\frac{1}{50}} \text{ Ans.}$$

2. $P(100 < X \leq 200) = F_X(200) - F_X(100)$

$$= \int_{100}^{200} f_X(x)\,dx$$

$$= \frac{1}{50} \cdot \int_{100}^{200} e^{-\frac{x}{50}}\,dx$$

$$= \frac{1}{50} \cdot 50 \int_2^4 e^{-y}\,dy$$

$$= e^{-2} - e^{-4} \approx 0.117$$

3. $P(X \leq 100) = F_X(100)$

$$= \int_{-\infty}^{100} f_X(x)\,dx$$

$$= \int_0^{100} \frac{1}{50} \cdot e^{-\frac{x}{50}}\,dx$$

$$= \int_0^2 e^{-y}\,dy$$

$$= 1 - e^{-2}$$

$$\approx 0.865$$