Optimal Estimation University of Florida
Mechanical and Aerospace Engineering

HW 3
Issued: September 11, 2009, Due: September 18, 2009 (in class)

Problem 1. [5 pt]
If $X \sim N(\mu, \sigma)$, compute $P(|X - \mu| > 3\sigma)$. If you need tabulated values of $erf(x)$, you can find them in the reading material available in the e-learning website (chapter 2)

Problem 2. [5+5 = 10 pt]
Show that if $Y \sim N(\mu, \sigma)$, then $\mathrm{E}[Y] = \mu$ and $var(Y) = \sigma^2$.

Problem 3. [10 pt] If X is a random variable that is uniformly distributed between a and b, and γ is a real constant, show that γX is uniformly distributed between γa and γb. (hint: define $Z \triangleq \gamma X$. Then $F_Z(z) = P(Z \leq z) = P(X \leq z/\gamma) = F_X(\dots) = ..$)

Problem 4. [10 pt]
Compute the mean and variance of a random variable X that is uniformly distributed between a and b.

Problem 5. [5 pt]
Show that for a r.v. X,

$$Var(X) = \mathrm{E}[X^2] - (\bar{X})^2. \quad [5 \text{ pt}]$$

Problem 6. [5+5 = 10 pt]
The unit square U in $\mathbb{R}^2$ is the region such that the points in this region have their x- and y-coordinates between 0 and 1. Let X be random vector that is uniformly distributed in the unit square. This means that

$$f_X(x) = \begin{cases} 1 & x \in U \quad \text{(note that } x \in \mathbb{R}^2) \\ 0 & \text{otherwise} \end{cases}$$

1. If $\mathcal{A}_1$ is a region that lies entirely in U, and its area is 0.53, what is the probability that $X \in \mathcal{A}_1$?
2. Let $\mathcal{A}_2$ be an arbitrary region in $\mathbb{R}^2$. Provide an expression for $P(X \in \mathcal{A}_2)$.

Problem 7 (Reading assignment, don't submit any work)**.** A Poisson distributed r.v. X is a discrete-type r.v. taking values in $0, 1, 2, \dots$, with the probability

$$P(X = k) = e^{-a}\frac{a^k}{k}, \tag{1}$$

where a is a fixed parameter. Read about Poisson r.v.'s from Chapter 1 of Stark and Woods.

Problem 8. [5+5 = 10 pts]

Compute the expected value and variance of a Poisson distributed random variable with parameter a.

Hint: the calculations involved can be simplified significantly when you use the following relationships, which are obtained by differentiating the Taylor series expansion of e^a with respect to a:

$$e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!}, \quad e^a = \frac{1}{a}\sum_{k=1}^{\infty} k\frac{a^k}{k!}, \text{ (differentiating once)}, \quad e^a = \frac{1}{a^2}\sum_{k=1}^{\infty} k(k-1)\frac{a^k}{k!} \text{ (differentiating twice)}$$

Problem 9. [20 pt]

Let X be the number of users of the wireless network `DigitalDowntown` in downtown Gainesville at any given time. X is modeled as a (discrete-type) r.v. that is Poisson distributed with parameter a. Suppose you are the manager of the ISP (Internet Service Provider) and have to decide how much bandwidth the routers to be installed should have. Suppose each user needs B MBPS of bandwidth, and you want your system to have enough bandwidth so that the probability of users' demand exceeding the available bandwidth is less than 0.01. How much bandwidth should your system have to satisfy these requirements? [10 pts] (Hint: use Tchebycheff Inequality)

Problems involving multiple random variables

Problem 10 (Response of linear systems to Gaussian inputs). [5 + 10 + 5 = 15 pt]

Show that

1. if $X \sim \mathcal{N}(\mu, \sigma^2)$ then aX is also Gaussian, where a is a constant. What is the mean and the variance of aX? (Hint: use linearity of expectation)
2. if $X \sim \mathcal{N}(0, \sigma^2)$ and $Y \sim \mathcal{N}(0, \sigma^2)$, and X and Y independent random variables, show that $X+Y \sim \mathcal{N}(0, 2\sigma^2)$. (Hint: use the following result that was not covered in class: The pdf of the sum $Z(= X+Y)$ of two independent r.v. X and Y is given by $f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)dy$.)
3. The problem above shows that a linear combination of two independent and Normally distributed r.v.'s is also Normal[1]. Now consider a discrete time linear system with scalar state and input

$$x_{k+1} = ax_k + bu_k,$$

where u_k is Normally distributed at every k, and u_k is independent of u_m whenever $m \neq k$ (you can think of u_k as a deterministic input corrupted by zero mean Gaussian noise that is due to the uncertainties in the actuator.) If the initial condition x_o is Gaussian and independent of u_k for all k, how is x_k distributed for an arbitrary k? (Provide precise arguments for your answer, but you don't have to provide parameters of the distribution, only the name)

Problem 11. [10 pt]

Show that for a random vector X,

$$Covar(X, X) = \mathrm{E}[XX^T] - \bar{X}\bar{X}^T. \quad [5 \text{ pt}]$$

[1]Although we proved it only for the case when both X and Y have the same mean and variance, the statement is true even when X and Y have different means and variances.