Optimal Estimation University of Florida
Mechanical and Aerospace Engineering

HW 5
Issued: September 25, 2009, Due: October 2, 2009 (in class)

Problem 1. [30 pt]

Consider the motion of a vehicle in a *straight line* that is equipped with a GPS receiver and an IMU (inertial measurement unit). The GPS unit provides *noisy* measurements of the absolute position with respect to a common origin, while the IMU unit provides *noisy* measurements of the relative position between two time instants. Specifically, if the vehicle gets a GPS measurement at time t second, the measurement is of the form $x(t) + \epsilon^{GPS}(t)$, where $x(t)$ is the position of the vehicle at time t and $\epsilon^{GPS}(t)$ is a measurement noise. If the vehicle gets an IMU measurement at time t seconds and the last IMU measurement it received was at τ seconds, then the IMU measurement is of the form $x(t) - x(\tau) + \epsilon^{IMU}(t - \tau)$, where ϵ^{IMU} is a measurement noise. Notice that all variables involved are scalars.

Assume that the first GPS measurement is obtained at time 0 second, and the first IMU measurement at time 0.5 second, the latter being a noisy measurement of $x(0.5) - x(0)$. The vehicle gets GPS measurements at 1 second intervals and IMU measurements at 0.5 second intervals.

1. Write down the measurements collected upto time= 3 seconds (*except* the GPS measurement collected at time= 3 second) in the form $\mathbf{z} = \mathbf{H}\boldsymbol{\theta} + \boldsymbol{\epsilon}$, where $\boldsymbol{\theta}$ is the vector of the unknown vehicle positions $x(0), x(0.5), x(1), \ldots, x(3)$ (i.e., positions at 0.5 second intervals) and $\mathbf{z}$ is the vector of the measurements, and $\boldsymbol{\epsilon}$ is the vector of measurement noises.

2. Assume that the measurement noises are 0 mean and every measurement is uncorrelated with every other. Choose a particular model of GPS and IMU unit (search online?) and decide on an appropriate covariance matrix of the noise vector $\boldsymbol{\epsilon}$ from the specifications provided by the manufacturers[1].

3. Generate a measurement vector $\mathbf{z}$ numerically in MATLAB assuming the vehicle moves at a speed of 80 kmph and starts from the position 500 m (which gives you the true positions), and using the `randn` function to generate the noise. Be careful that the generated noise should have the variance that you have chosen for your sensors.

4. Compute the BLUE estimate of $\boldsymbol{\theta}$ for this particular measurement vector.

5. Repeat the part above for a different measurement vector, by using a distinct noise vector, and calculate the positions again. Repeat this process 100 times and plot the resulting estimates against the true positions.

[1] you're allowed to choose any make/model you want, but choose something for which the vendor provides enough information for you to decide on the covariance matrix.

Problem 2. [20 pt]

Repeat the parts 1,2,3,4 of the problem above when the the measurement vector includes the GPS measurement collected at time= 3 second. In doing the numerical calculation, take care to ensure that the measurements that are common between the two problems have the same numerical values[2].

Does the estimate of the positions change from that obtained in part 4 of the previous problem?

Problem 3. [10 pt]

Consider the problem of fitting a straight line to m data pairs $(x_i, y_i), i = 1, \dots, m$. Start with the model $y_i - \bar{y} = a(x_i - \bar{x}) + b + \varepsilon_i$, where $\bar{x} = \frac{1}{m}(x_1 + \cdots + x_m)$, $\bar{y} = \frac{1}{m}(y_1 + \cdots + y_m)$. Show that the least squares fit to the parameters a and b are given by

$$\hat{a} = \frac{\sum_{i=1}^{m}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{m}(x_i - \bar{x})^2}, \quad \hat{b} = 0. \qquad [\ 10\ \text{pt}]$$

Problem 4. Let $\theta \in \mathbb{R}^n$ and $\hat{\theta}^*$ be the best linear unbiased estimate of θ. Show that

1. $\hat{\theta}^*$ minimizes $\mathrm{E}[(\theta - \hat{\theta})^T(\theta - \hat{\theta})]$ over all linear unbiased estimates $\hat{\theta}$, i.e., if $\hat{\theta}$ is a linear unbiased estimate of θ, then

$$\mathrm{E}[(\theta - \hat{\theta}^*)^T(\theta - \hat{\theta}^*)] \leq \mathrm{E}[(\theta - \hat{\theta})^T(\theta - \hat{\theta})]. \qquad [10\ \text{pt}]$$

2. $L\hat{\theta}^*$ is the best linear unbiased estimate of $L\theta$, where L is a fixed matrix with compatible dimensions. [10 pt]

3. $\hat{\theta}^*$ minimizes $\mathrm{E}[(\theta - \hat{\theta})^T P(\theta - \hat{\theta})]$ for any non-negative definite matrix P over all linear unbiased estimates $\hat{\theta}$. That is, if $\hat{\theta}$ is a linear unbiased estimate of θ, then

$$\mathrm{E}[(\theta - \hat{\theta}^*)^T P(\theta - \hat{\theta}^*)] \leq \mathrm{E}[(\theta - \hat{\theta})^T P(\theta - \hat{\theta})]. \qquad 10\ \text{pt}$$

   (Hint: use the result established in part 2)

4. $\hat{\theta}^*$ minimizes the error covariance matrix $\mathrm{E}[(\theta - \hat{\theta})(\theta - \hat{\theta})^T]$ over all linear unbiased estimates $\hat{\theta}$ in the usual matrix ordering sense, i.e.,

$$\mathrm{E}[(\theta - \hat{\theta}^*)(\theta - \hat{\theta}^*)^T] \leq \mathrm{E}[(\theta - \hat{\theta})(\theta - \hat{\theta})^T]$$

   for every $\hat{\theta}$ that is a linear unbiased estimate of θ. Remember that for two matrices A and B, $A \leq B$ means $B - A$ is positive semidefinite, or equivalently, $A - B$ is negative semidefinite.[10 pt]

[2] Be careful. Since `randn` is involved, if you rerun your MATLAB script to generate the measurements for this problem the "common" measurements between the two problems will have different numerical values