Solution to HW 6

EML 6934

P1 $X \sim N(\mu, \sigma^2)$

a single observation of X, say x_0, is given

We want to compute $\hat{\mu}_{ML}$ & $\hat{\sigma}^2_{ML}$ based on x_0

$$f_X(x;\theta) = \frac{1}{\sqrt{2\pi\theta_2}} \cdot e^{-\frac{(x_0-\theta_1)^2}{2\theta_2}} \qquad \text{calling} \quad \begin{array}{l}\theta_1 = \mu \\ \theta_2 = \sigma^2\end{array}$$

$$\Rightarrow \ell(\theta; x_0) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x_0-\theta_1)^2}{2\theta_2}}$$

$$\Rightarrow L(\theta; x_0) = \log \ell(\theta; x_0) = \log\frac{1}{\sqrt{2\pi}} - \frac{1}{2}\log\theta_2 - \frac{(x_0-\theta_1)^2}{2\theta_2}$$

$$\therefore \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{\partial L}{\partial \theta_1} = 0 \quad \text{— (1)}$$

$$\frac{\partial L}{\partial \theta_2} = 0 \quad \text{— (2)}$$

$$(1) \Rightarrow -\frac{1}{2\theta_2}2(x_0-\theta_1)(-1) = 0 \Rightarrow \theta_1 = x_0 \Rightarrow \hat{\mu}_{ML} = x_0 \quad \text{Ans}$$

$$(2) \Rightarrow +\frac{1}{2}\frac{1}{\theta_2}(-1) + \frac{1}{2}(x_0-\theta_1)^2(+1)\frac{1}{\theta_2^2} = 0$$

$$\Rightarrow \theta_2 = (x_0-\theta_1)^2 = 0 \quad (\text{from } \theta_1 = x_0)$$

$$\Rightarrow \hat{\sigma}^2_{ML} = 0 \quad \text{Ans}$$

estimate

$\hat{\mu}_{ML} = X$

estimator !

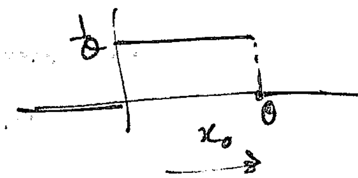
$$E[\hat{\mu}_{ML}(X)] = E[X] = \mu \quad : \Rightarrow \text{unbiased}$$

$$E[\hat{\sigma}^2_{ML}(X)] = E[0] = 0 \neq \sigma^2 \Rightarrow \text{biased} \quad (\text{unless } \sigma^2 = 0\,!)$$

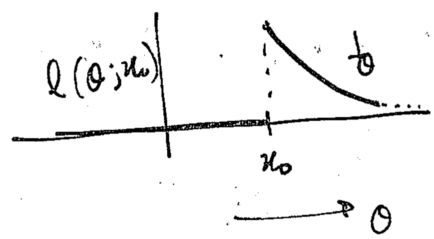
P2 X is uniformly distributed between 0 and θ

a single observation x_0 is obtained

$$f_X(x_0;\theta) = \frac{1}{\theta} \; ; \; 0 < x_0 \leq \theta$$
$$= 0 \qquad x_0 < 0 \text{ or } x_0 > \theta$$



$$\Rightarrow l(\theta; x_0) = \frac{1}{\theta} \qquad \theta \geq x_0$$
$$0 \qquad \text{o/w}, \; (\text{i.e., } \theta < x_0)$$



$\therefore \hat{\theta}_{ML}$ = that value of θ that maximizes $l(\theta; x_0)$

$= x_0$ from the definition of $l(\theta)$

$\Rightarrow \hat{\theta}_{ML} = x_0$ Ans. $\leftarrow$ estimate.

$\hat{\theta}_{ML}(X = x_0) = x_0$

$\Rightarrow \hat{\theta}_{ML}(X) = X \leftarrow$ estimator!

$\therefore E[\hat{\theta}_{ML}] = E[X] = \frac{\theta}{2} \neq \theta \Rightarrow$ Biased!

P3 If there were only observation which is, say, k_0, then $l(a; k_0) = \dfrac{e^{-a} a^{k_0}}{k_0!}$

With n-observations, say $k_1, k_2 \ldots k_n$, we have

$$l(a; k_1, k_2, \ldots k_n) = \left(\frac{e^{-a} a^{k_1}}{k_1!}\right)\left(\frac{e^{-a} a^{k_2}}{k_2!}\right) \cdots \left(\frac{e^{-a} a^{k_n}}{k_n!}\right)$$

$$\Rightarrow L(a; k_1, \ldots k_n) = \log l(a; k_1 \ldots k_n)$$

$$= -na + (k_1 + k_2 + \cdots k_n) \log a - \sum_i \log (k_i)!$$

to find $\hat{a}_{ML}$, set $\dfrac{\partial L}{\partial a} = 0$

$$\Rightarrow -n + \frac{\sum k_i}{a} - 0 = 0$$

$$\Rightarrow a = \frac{\sum k_i}{n}$$

$$\Rightarrow \hat{a}_{ML} = \frac{\sum_{i=1}^{n} k_i}{n} \text{ — estimate}$$

Ans

Estimator : $\hat{a}_{ML} = \dfrac{\sum_{i=1}^{n} X_i}{n}$

$$\Rightarrow E[\hat{a}_{ML}] = \frac{1}{n} \sum_{i=1}^{n} E[X_i] \quad \text{(linearity of expectation)}$$

$$= \frac{1}{n} \cdot na$$

$$= a$$

$\Rightarrow$ Unbiased !

P4 : denote the n samples of X as $x_1, x_2, \cdots x_n$

$$\Rightarrow \ell(\lambda; x_1, \cdots x_n) = \left(\frac{\lambda}{2} e^{-\lambda|x_1|}\right)\left(\frac{\lambda}{2} e^{-\lambda|x_2|}\right)\cdots\left(\frac{\lambda}{2} e^{-\lambda|x_n|}\right)$$

$$\Rightarrow L(\lambda; x_1, \cdots x_n) = \log \ell(\lambda; x_1 \cdots x_n)$$

$$\propto n \log \lambda - \lambda \sum_{i=1}^{n} |x_i|$$

to find $\hat{\lambda}_{ML}$, set $\frac{\partial L}{\partial \lambda} = 0 \Rightarrow \frac{n}{\lambda} - \sum_{i=1}^{n} |x_i| = 0$

$$\Rightarrow \lambda = \frac{n}{\sum_{i=1}^{n} |x_i|}$$

$$\Rightarrow \hat{\lambda}_{ML} = \frac{n}{\sum_{i=1}^{n} |x_i|} \quad \text{— estimate}$$

Ans

$$\hat{\lambda}_{ML} = \frac{n}{\sum_{i=1}^{n} |X_i|} \quad : \text{estimator}$$

to check unbiasedness, we'll have to evaluate

$$E[\hat{\lambda}_{ML}] = n\, E\left[\frac{1}{\sum_{i=1}^{n} |X_i|}\right]$$

assuming X_i are independent and Laplacian distributed

This is quite complicated so omitted!