Error Amplification and Disturbance Propagation in Vehicle Strings with Decentralized Linear Control

Prabir Barooah and João P. Hespanha
Dept. of Electrical and Computer Eng., Univ. of California, Santa Barbara, CA 93106
Email:{pbarooah,hespanha}@ece.ucsb.edu

Abstract— **We consider the problem of controlling a string of vehicles moving in one dimension so that they all follow a lead vehicle with a constant spacing between successive vehicles. We examine the symmetric bidirectional control architecture, where the control action on a vehicle depends equally on the spacing errors with respect to its adjacent vehicles. Performance of this decentralized scheme in terms of spacing error amplification and disturbance propagation is investigated. The results established in this paper show that a symmetric bidirectional architecture with a linear controller suffers from fundamental limitations on closed loop performance that cannot be mitigated by appropriate control design.**

I. Introduction

We consider the problem of controlling a string of vehicles moving in one dimension so that they all follow a lead vehicle with a constant spacing between successive vehicles (c.f. Figure 1). Due to its relevance to developing automated highway systems, this problem has been studied in several recent papers [2, 3, 5, 6, 8].

A control architecture investigated in the literature is *predecessor following* – where the control action on a particular vehicle depends on its spacing error with the predecessor, i.e., the vehicle in front of it. This scheme is decentralized, since every vehicle can compute its control action based purely on information it can measure with on-board sensors. In a recent paper by Seiler *et. al.* [5] it was shown that if the vehicle model, denoted by $H(s)$, has a double integrator, then a predecessor following control architecture will lead to *string instability* [7] for any dynamic compensator $K(s)$. That is, spacing errors will get amplified along the string of vehicles. They also showed that string stability can be restored if a *predecessor and leader following* control architecture is used, where the control action on a particular vehicle is based on the predecessor's position as well as the lead vehicle's position. To implement such a scheme, lead vehicle position needs to be communicated to all the vehicles in the platoon. Khatir et. al. [3] also considered the predecessor following control scheme and showed that string stability is not possible for a certain class of plants and controllers.

Another control architecture investigated in the literature, and on which we focus in this paper, is *bidirectional control*. In this scheme, the control action on a particular vehicle depends on the spacing errors with respect to its predecessor and its follower. Most human drivers use information about preceding and following vehicles to control their own vehicles, so bidirectional control is intuitively appealing. Seiler

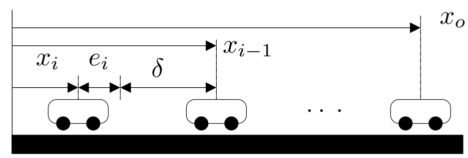

Fig. 1. platoon of vehicles

et. al. [5] considered *symmetric bidirectional control*, where *symmetric* refers to the control effort being equally dependent on the spacing errors with the preceding vehicle and the following vehicle. They showed that, provided the controller does not have an integrator, such a control scheme is not scalable for disturbance rejection. That is, it is impossible to get a uniform bound on the $\mathcal{H}_\infty$-norm of the transfer function from the disturbances acting on the vehicles to the spacing errors as the platoon size increases. However, the question of whether or not, in the absence of disturbances, the spacing errors between vehicles will get amplified along the string was left unanswered.

The question of string stability or error amplification for the 1-D vehicular platoon control problem has been investigated in various forms by several researchers. Conclusive results about error amplification have have been established for predecessor-following and predecessor-and-leader-following schemes [3, 5]. However, such results were lacking for the bidirectional scheme, both on spacing error amplification and disturbance propagation. We fill this gap by establishing results on error amplification and providing an extension of the result by Seiler *et. al.* [5] on disturbance propagation.

The main results of this paper are the following. We show that with symmetric bidirectional control, if the total number of integrators between the plant and the controller is more than two, then the closed loop platoon error dynamics will be unstable for a sufficiently large number of vehicles (theorem 1). If $H(s)K(s)$ has two integrators, then the steady state spacing errors all converge to 0 no matter how large the platoon is, provided the lead vehicle moves at a constant velocity (theorem 2). However, if the lead vehicle trajectory deviates from a constant-velocity one, the $\mathcal{L}_2$ norm of the spacing errors will grow unbounded as the number of vehicles increases, even if the deviation has bounded $\mathcal{L}_2$-norm (theorem 3). On the other hand, if $H(s)K(s)$ has only one integrator and the lead vehicle moves at a constant

Preprint submitted to 44th IEEE Conference on Decision and Control and European Control Conference ECC 2005. Received .

velocity, the steady state error is finite for a finite platoon size, but the norm of this error grows without bound as the number of vehicles in the platoon increases (theorem 2). When $H(s)K(s)$ has one integrator, if the deviation of the lead vehicle's trajectory from a constant velocity one is $\mathcal{L}_2$-norm bounded, then the spacing errors are $\mathcal{L}_2$ norm bounded, too, irrespective of the number of vehicles.

Moreover, when $H(s)K(s)$ has two integrators, the $\mathcal{H}_\infty$ norm of the transfer function from the disturbances acting on the vehicles to the spacing errors will grow without bound as the number of vehicles increases (theorem 4). Thus, even if the lead vehicle is moving at constant velocity, if disturbances are present in the control signal – as they invariably will – large spacing errors will result.

The case of $H(s)K(s)$ having no integrators is not considered, since a realistic model of a vehicle for a highway will have at least one integrator. In fact, vehicle models with double integrators are quite common in the literature, as a double integrator follows directly from feedback linearization [5, 6, 8].

Past research on string stability in 1-D vehicular platoon has mostly focused on the predecessor-following and predecessor-and-leader-following architectures. In those cases, spacing errors propagate in one direction, from one vehicle to the next. The question of string stability/error amplification is usually answered by looking at the transfer functions that relate the spacing errors between two successive vehicle pairs. However, in a bidirectional control, errors "propagate" in both ways. It is still important to know if small spacing errors between two vehicles can result in large spacing errors among other vehicle pairs in the string. However, other approaches and techniques are required to pose and answer this question. Seiler *et. al.* [5] used the transfer function from the disturbances to the spacing errors in a bidirectional control, and showed that the $\mathcal{H}_\infty$ norm of this transfer function grows without bound as the number of vehicles increases, provided the $H(s)$ has two integrators and $K(s)$ has none. We follow a similar approach, but look at the transfer functions relating both lead vehicle position and disturbances to spacing errors. Our results show that irrespective of the number of integrators in the plant or the controller, linear control of a vehicular platoon with a symmetric bidirectional architecture suffers from limitations that makes it fundamentally impossible to design a controller that achieves good closed loop performance for arbitrarily large platoons.

Certain graph-theoretic tools are utilized in establishing these results. The control architecture for the vehicular platoon can be represented by a graph whose nodes correspond to vehicles and edges to relative position measurements that a vehicle's controller uses. It is known that the eigenvalues of the Laplacian of this graph play a key role in the stability of the platoon [1]. Our analysis makes use of the distribution of eigenvalues of the Laplacian of this graph.

We consider all the vehicles to have identical dynamics and identical controllers, as in [5]. Khatir et. al. [3] showed that this problem can be mitigated somewhat with non-identical controllers – the spacing errors can now be uniformly bounded but at the cost of the velocities becoming unbounded as the number of vehicles increases.

The rest of the paper is organized as follows. In section II, we formulate the problem and derive the closed loop transfer functions for symmetric bidirectional control. In section III, we establish certain necessary conditions for the closed loop error dynamics to be stable with symmetric bidirectional control. In IV we examine the performance of the symmetric bidirectional control when the lead vehicle moves at a constant velocity. In section V, we investigate spacing error amplification when the lead vehicle velocity is not constant. In section VI, we consider the effect of disturbances in control signals on the spacing errors. The paper concludes with a summary and discussion of future research directions.

II. Error Dynamics in Symmetric Bidirectional Control

Our problem formulation is essentially the same as that of [5]. $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural, real and complex numbers, respectively. We consider a string of $N+1$ vehicles moving in one dimension (Fig. 1). Let $x_0(t)$ denote the position of the lead vehicle and $x_i(t),\ i \in \{1, 2, \ldots, N\}$, the position of the ith follower vehicle. The spacing error of the ith vehicle is defined by

$$e_i(t) = x_{i-1}(t) - x_i(t) - \delta,$$

where the desired spacing, δ, is a positive constant. The control objective is to keep the spacing error for every vehicle as small as possible while maintaining closed loop stability. We assume that,

1) All vehicles have the same model, denoted by $H(s)$,
2) $H(s)$ is linear, SISO, and it has at least one integrator,
3) All vehicles use the same control law, and
4) the string of vehicles start with zero spacing errors, from rest, and the lead vehicle starts at $x_0(0) = 0$. Hence, $x_i(0) = -i\delta$.

Let $X(s)$ denote the Laplace transform of a time-domain signal $x(t)$: $X(s) := \mathcal{L}(x(t))$. Applying the assumptions, each vehicle can be modelled in the Laplace domain as

$$X_i(s) = H(s)(U_i(s) + D_i(s)) + \frac{x_i(0)}{s}, \quad 1 \le i \le N, \tag{1}$$

where $x_i(0)$ is the initial position of the ith vehicle, $U_i(s)$ is the Laplace transform of the control signal and $D_i(s) = \mathcal{L}(d_i(t))$ is the Laplace transform of the input disturbance $d_i(t)$ to the ith vehicle. The ith spacing error in the Laplace domain is given by

$$E_i(s) = X_{i-1}(s) - X_i(s) - \frac{\delta}{s}, \quad 1 \le i \le N. \tag{2}$$

Using (2) and (1) and applying the fourth assumption, we can write the error dynamics of the entire vehicle platoon as

$$\bar{E}(s) = X_0(s)\phi_1 + P(s)\left[\bar{D}(s) + \bar{U}(s)\right] \tag{3}$$

where $\phi_1 \in \mathbb{R}^N$ is the 1st element of the canonical basis of $\mathbb{R}^N$ and

$$\begin{aligned}
&\bar{E}(s) := \mathcal{L}(\bar{e}(t)), \quad \bar{e}(t) = [e_1(t) \dots e_N(t)]^T, \\
&\bar{D}(s) := \mathcal{L}(\bar{d}(t)), \quad \bar{d}(t) = [d_1(t) \dots d_N(t)]^T, \\
&\bar{U}(s) := [U_1(s) \dots U_N(s)]^T, \\
&P(s) := -H(s)M^T,
\end{aligned}$$

where M is defined as

$$M := \begin{bmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & 1 \end{bmatrix} \in \mathbb{R}^{N \times N}. \tag{4}$$

In a bidirectional control scheme, each vehicle bases its control action on the error feedback from its predecessor and follower. The control action is

$$U_i(s) = K(s)\left(E_i(s) - E_{i+1}(s)\right), \quad 1 \leq i \leq N. \tag{5}$$

Since the last vehicle in the string does not have a follower, it uses the controller $U_N(s) = K(s)E_N(s)$. The vector of platoon control inputs is given by

$$\bar{U}(s) = K(s)M\bar{E}(s)$$

which is a restatement of (5). Eliminating $\bar{U}(s)$ from (3) we can write the closed-loop error dynamics of the platoon, which we present next.

Lemma 1. *The closed-loop dynamics of the platoon spacing errors is given by*

$$\bar{E}(s) = G_{x_oe}(s)X_o(s) + G_{de}(s)\bar{D}(s). \tag{6}$$

where

$$\begin{aligned}
G_{x_oe}(s) &= [I + H(s)K(s)L]^{-1}\,\phi_1, &(7)\\
G_{de}(s) &= -H(s)\,[I + H(s)K(s)L]^{-1}\,M^T, &(8)
\end{aligned}$$

and $L := M^TM \in \mathbb{R}^{N\times N}$. □

The matrix $L \in \mathbb{R}^{N\times N}$ is given by

$$L := M^TM = \begin{bmatrix} 1 & -1 & 0 & \dots & & \\ -1 & 2 & -1 & \dots & & \\ 0 & -1 & 2 & -1 & \dots & \\ & & & & \ddots & -1 \\ & & & & -1 & 2 \end{bmatrix}. \tag{9}$$

A control architecture for the vehicular platoon can be represented by an undirected graph whose nodes correspond to vehicles and edges to relative position measurements that a vehicle's controller uses. We call it the *control architecture graph*. In symmetric bidirectional control, each vehicle uses relative position measurements with its predecessor and its follower (except for the first and the last vehicles). In the control architecture graph, this corresponds to each node having an edge with two other nodes (c.f. Figure 2). The matrix L is a sub-matrix of the Laplacian of the control architecture graph for the symmetric bidirectional control scheme, and is obtained by removing the row and the column corresponding to the lead vehicle. We now establish certain useful properties of L that will be used later.

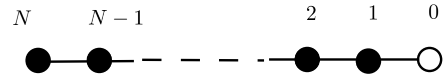

Fig. 2. The control architecture graph for bidirectional control.

Lemma 2. *consider the matrix L, as defined in* (9)*. Let* λ_{min} *be the smallest eigenvalue of L, and let* $u_1 := [u_{11}, u_{21}, ..., u_{N1}]^T$ *be a unit-norm eigenvector of L corresponding to* λ_{min}*. Then the following are true:*

1) $\exists \alpha_1,\, \alpha_2 > 0$ *such that*
$$\alpha_1/N^2 < \lambda_{min} \leq \alpha_2/N^2, \quad \forall N.$$
2) $|u_{11}| > N^{-1/2}$.

Proof: To proof the first statement, we start off by proving that L is positive definite. Consider the product x^TLx, where $x = [x_1, x_2, \dots, x_N]^T \in \mathbb{R}^N$. Then, $x^TLx = 2\sum_{i=1}^N x_i^2 - 2\left(x_1x_2 + x_2x_3 + \dots + x_{N-2}x_{N-1}\right) = \sum_{i=1}^{N-1}(x_i - x_{i+1})^2 + x_1^2 + x_N^2 > 0$. This proves that L is positive definite. Since $L = L^T$, all eigenvalues of L^{-1} are positive real. So the smallest eigenvalue of L is the inverse of the largest eigenvalue of L^{-1}. Note that L^{-1} is given by

$$L^{-1} = \begin{bmatrix} N & N-1 & \dots & 2 & 1 \\ N-1 & N-1 & \dots & 2 & 1 \\ & & \ddots & & \\ 2 & 2 & \dots & 2 & 1 \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix}$$

To prove it, simply multiply the matrix with L and check that an identity matrix results. From Gerschgorin circle theory, we know that an upper bound for the largest eigenvalue of L^{-1} is $\sum(1+2+\dots+N) < N^2$. Therefore, a lower bound for the smallest eigenvalue of L is $1/N^2$. That is, $\lambda_{min} > 1/N^2$. This proves the lower bound on the minimum eigenvalue with $\alpha_1 = 1$. To get the upper bound on λ_{min}, let us write L as

$$L = \left[\begin{array}{c|c} 1 & -\phi_{1(N-1)}^T \\ \hline \phi_{1(N-1)} & L_1 \end{array}\right]_{N\times N}$$

where $\phi_{1(N-1)}$ is the first element of the canonical basis vector of $\mathbb{R}^{N-1}$ and $L_1 \in \mathbb{R}^{N-1\times N-1}$ is the so-called finite-difference matrix:

$$L_1 := \begin{bmatrix} 2 & -1 & 0 & \dots & & \\ -1 & 2 & -1 & \dots & & \\ 0 & -1 & 2 & -1 & & \\ \vdots & & & & \ddots & -1 \\ & & & & -1 & 2 \end{bmatrix}_{N-1\times N-1}.$$

From Cauchy's Interlacing Theorem, we know that $\lambda_{min} \leq \mu_{min}$, where μ_{min} is the smallest eigenvalue of L_1. It is known [4] that $\mu_{min} = 4\sin^2(\pi/2N)$. Moreover, for $\theta > 0$, $\sin\theta \leq \theta$. Hence, $\mu_{min} \leq \pi^2/N^2$, which establishes the upper bound on λ_{min} with $\alpha_2 = \pi^2$.

To prove the second statement, note that since u_1 is an eigenvector of L corresponding to the smallest eigenvalue of L, u_1 is also an eigenvector of L^{-1} corresponding to its largest eigenvalue. Since L^{-1} is a positive matrix, Perron-Frobenius theory tells us that $|u_1|$ is also an eigenvector of L^{-1} corresponding to its largest eigenvalue and that $|u_1|$ is a positive vector. Thus, we can make the unit-norm eigenvector

Preprint submitted to 44th IEEE Conference on Decision and Control
and European Control Conference ECC 2005. Received .

u_1 of L, corresponding to λ_{min}, consist entirely of positive numbers. Let's write down the equation $Lu_1 = \lambda_{min} u_1$ in expanded form:

$$\begin{bmatrix} u_{11} - u_{21} \\ -u_{11} + 2u_{21} - u_{31} \\ -u_{21} + 2u_{31} - u_{41} \\ \dots \\ \dots \end{bmatrix} = \begin{bmatrix} \lambda_{min} u_{11} \\ \lambda_{min} u_{21} \\ \lambda_{min} u_{31} \\ \dots \\ \dots \end{bmatrix}$$

It is easy to check from these equations and the positivity of u_{i1}'s that u_{i1}'s form a decreasing sequence: $u_{11} > u_{21} > \dots u_{N1} > 0$. Since $\sum u_{i1}^2 = 1$, it follows that $u_{11}^2 > 1/N$. This proves the Lemma. ■

III. Closed Loop Stability with Symmetric Bidirectional Control

We have already discussed that for $H(s)$ to be a reasonable model of a vehicle in a highway, $H(s)$ must have at least one integrator. In this section we establish that with symmetric bidirectional control, for closed loop stability of the platoon with arbitrary N, $H(s)K(s)$ cannot have more than two integrators. Along with this main result, we also establish certain other properties that are required for closed loop stability. These will be used in the subsequent sections.

Theorem 1. *Consider the closed loop error dynamics of the platoon with symmetric bidirectional control, given by* (6)*.*

1) *For closed loop stability of the platoon with arbitrary N, $H(s)K(s)$ cannot have more than two integrators.*
2) *For closed loop stability of the platoon with N vehicles following the leader, every transfer function $G_i(s) = 1/(1 + \lambda_i H(s)K(s))$, $i = \{1, 2, \dots, N\}$ must be stable, where λ_i is the ith eigenvalue of the matrix $L \in \mathbb{R}^{N \times N}$ defined in* (9)*, and consequently, $K(s)$ cannot have zeros at* 0*.*
3) *If we define $H(s)K(s) = C(s)/s^k$ with $C(0)$ is finite, then for closed loop stability with arbitrary N, $C(0) >$* 0*.* □

Proof: We start by proving the second statement, which follows from the results established by Fax *et. al.* [1]. It is also easy to see once we simplify equation (7). Since L is symmetric, $\exists U \in \mathbb{R}^{N \times N}$ with $U^T U = UU^T = I$ s.t. $L = U \Lambda U^T$ where Λ is a real diagonal matrix containing the eigenvalues of L and $U = [u_1, u_2, \dots, u_N]$, u_i being a unit-norm eigenvector of L corresponding to the ith eigenvalue. The eigenvalues are arranged as

$$\lambda_{min} \leq \lambda_2 \leq \dots \leq \lambda_{N-1} \leq \lambda_{max}.$$

Hence,

$$\begin{aligned} I + H(s)K(s)L &= U(I + H(s)K(s)\Lambda)U^T \\ \Rightarrow (I + H(s)K(s)L)^{-1} &= U\left(I + (HK)\Lambda\right)^{-1} U^T. \end{aligned}$$

Using the above and (7), we get

$$\begin{aligned} G_{x_o e}(s) &= U\left(I + H(s)K(s)\Lambda\right)^{-1} U^T \phi_1 \\ &= U \Psi(s) U^T \phi_1, \end{aligned} \tag{10}$$

where the matrix $\Psi(s) \in \mathbb{C}^{N \times N}$ is defined as

$$\begin{aligned} \Psi(s) &:= (I + H(s)K(s)\Lambda)^{-1} \\ &= \begin{bmatrix} \frac{1}{1+\lambda_{min} H(s)K(s)} & & \\ & \ddots & \\ & & \frac{1}{1+\lambda_{max} H(s)K(s)} \end{bmatrix}. \end{aligned} \tag{11}$$

This gives us, with (10), that

$$G_{x_o e}(s) = \begin{bmatrix} \sum_{i=1}^N \frac{1}{1 + \lambda_i H(s)K(s)} u_{1i}^2 \\ \sum_{i=1}^N \frac{1}{1 + \lambda_i H(s)K(s)} u_{1i} u_{2i} \\ \dots \\ \sum_{i=1}^N \frac{1}{1 + \lambda_i H(s)K(s)} u_{1i} u_{Ni} \end{bmatrix} \tag{12}$$

It is clear now that for the closed loop to be stable, each of the transfer functions $1/(1+\lambda_i H(s)K(s))$ for $i \in \{1, 2, \dots, N\}$ must be stable. As a consequence, there cannot be any unstable pole zero cancellation between $H(s)$ and $K(s)$. Since $H(s)$ has at least one integrator by assumption, $K(s)$ cannot have any zeros at 0.

To prove (1), we consider the root locus of the system $H(s)K(s)$. Suppose $H(s)K(s)$ has three integrators. Then, at least one of the branches of the root loci will depart to the right half plane even for an arbitrarily small gain. Since λ_{min} can be arbitrarily small for N arbitrarily large (lemma 2), this means that $1/(1 + \lambda_{min} H(s)K(s))$ will be unstable for a large enough N. Thus, $H(s)K(s)$ cannot have three integrators. It must be a well known fact that for a plant $G(s)$ to be stabilizable by a real gain in negative feedback, it cannot have three integrators, though we were unable to find any reference for it. The extension of these arguments to the case of more than three integrators is trivial.

To prove (3), let $C(s) = N_c(s)/D_c(s)$ where $N_c(s)$ and $D_c(s)$ are coprime polynomials. From the above, $C(s)$ cannot have zeros at the origin. Therefore $C(s)$ does not have poles or zeros at the origin, so $N_c(0) \neq 0$ and $D_c(0) \neq 0$. Consider the case when $H(s)K(s)$ has two integrators, so the characteristic polynomial of $1/(1 + \lambda_{min} H(s)K(s))$ is $s^2 D_c(s) + \lambda_{min} N_c(s)$. If $N_c(0) < 0$, then $\lambda_{min} N_c(0) < 0$ and the closed loop will have at least one unstable pole. Thus $N_c(0) > 0$. The coefficient of s^2 in the characteristic polynomial is $D_c(0) + \lambda_{min} c_2$, where c_2 is the coefficient of s^2 in $N_c(s)$. If $D_c(0) < 0$ the coefficient of s^2 will be negative when λ_{min} is small enough, i.e., for a large enough N, even when c_2 is positive. This will make the closed loop unstable. Thus, in order to have closed loop stability for arbitrary N, we must have $D_c(0) > 0$. Hence, $C(0) > 0$. These arguments can be repeated for the case when $H(s)K(s)$ has one integrator, and we arrive at the same result. This proves the theorem. ■

IV. Control Performance When Lead Vehicle Moves with a Constant Velocity

In an automated highway system, in general the lead vehicle may be expected to move at a constant velocity. In that case $x_o^{\text{ref}}(t) = v_o t$, where v_o is the desired constant velocity. In this section we will show that if the lead vehicle moves at a constant velocity, i.e.,

$$x_o(t) = x_o^{\text{ref}}(t) = v_o t, \tag{13}$$

and $H(s)K(s)$ has two integrators, then all the platoon spacing errors can be made to converge to 0. If $H(s)K(s)$ has a single integrator, then the steady state platoon spacing error vector is non-zero, and the norm of the steady-state error grows without bound as N increases.

It is reasonable to expect that the lead vehicle will not be able to move at a constant velocity and there will deviations from the nominal value. One thus needs to study the effects of perturbation to (13). We leave that for the next section.

Theorem 2. *Consider the case when there are no disturbances acting on the vehicles, i.e., $\bar{d}(t) \equiv 0$, and the lead vehicle moves at a constant velocity, i.e., $x_o(t) = v_o t$, where $v_o > 0$ is the desired velocity. Let $K(s)$ be such that it achieves closed loop stability of the platoon error dynamics for arbitrary N with symmetric bidirectional control. Then the following are true:*

1) *If $H(s)K(s)$ has two integrators, then, $\forall\, N \in \mathbb{N}$,*

$$\lim_{t\to\infty} \bar{e}(t) = 0.$$

2) *If $H(s)K(s)$ has one integrator, then for a platoon of size N, $\exists\, e_\infty \in \mathbb{R}^N$, such that,*

$$\lim_{t\to\infty} \bar{e}(t) = e_\infty \neq 0,$$

and, for every $R > 0$, $\exists N_o \in \mathbb{N}$ such that $||e_\infty||_2 > R, \quad \forall\, N > N_o$. □

Proof: When $H(s)K(s)$ has a double integrator, we can represent $H(s)K(s)$ as $C(s)/s^2$, where $C(s)$ does not have poles or zeros at zero and $C(0) > 0$. This follows from theorem 1. Consider the spacing error of the kth vehicle. Since $x_o(t) = v_o t$, so $X_o(s) = v_o/s^2$. This, together with equations (12) and (6) gives us

$$sE_k(s) = \sum_{i=1}^{N} \frac{sv_o}{s^2 + \lambda_i C(s)} u_{1i} u_{ki}$$

Since $K(s)$ stabilizes the platoon dynamics, $1/(s^2+\lambda_i C(s))$ is a stable transfer function for $i \in \{1, 2, \dots, N\}$ and therefore $\lim_{s\to 0} sv_o/(s^2 + \lambda_i C(s)) = 0$. Since u_{ij}'s are bounded numbers, $sE_k(s) \to 0$ as $s \to 0$. Hence, from the Final Value Theorem,

$$\lim_{t\to\infty} \bar{e}(t) = \lim_{s\to 0} s\bar{E}(s) = 0.$$

This proves the first statement of the theorem.

Now we consider the case of $H(s)K(s)$ having only one integrator. We can represent $H(s)K(s)$ as $C(s)/s$ where $C(s)$ doesn't have poles or zeros at the origin and $C(0) > 0$ (theorem 1). Since $Xo(s) = v_o/s^2$, we have

$$\begin{aligned} s\bar{E}(s) &= Us\Psi(s)\frac{v_o}{s^2}U'\phi_1 \\ &= UQ(s)U'\phi_1, \end{aligned}$$

where $Q(s)$ is defined as

$$Q(s) := \begin{bmatrix} \frac{v_o}{s+\lambda_{min}C(s)} & & \\ & \ddots & \\ & & \frac{v_o}{s+\lambda_{max}C(s)} \end{bmatrix}.$$

Hence, once again from the Final Value Theorem,

$$\lim_{t\to\infty} \bar{e}(t) = \lim_{s\to 0} s\bar{E}(s) = UQ(0)U'\phi_1 := e_\infty,$$

which is a constant vector. Thus the steady state error converges to a constant vector.

To prove that e_∞ grows unbounded with N, note that

$$||\bar{e}_\infty||_2^2 = \phi_1^T UQ(0)^T Q(0)U^T\phi_1. \tag{14}$$

Since $U^T\phi_1$ is the first row of U and $Q(0)$ is a real diagonal matrix, we can reduce (14) to

$$\begin{aligned} ||\bar{e}_\infty||_2 &= \left(\sum_{i=1}^{N}\left(\frac{v_o}{\lambda_i C(0)}\right)^2 u_{1i}^2\right)^{1/2} \\ &> \frac{v_o}{\lambda_{min}C(0)}|u_{11}|. \end{aligned}$$

From lemma 2, we have $1/\lambda_{min} > N^2/\alpha_2$ and $|u_{11}| > 1/N^{1/2}$. Using these in the above, we get

$$||\bar{e}_\infty||_2 > \gamma N^{3/2},$$

where $\gamma = v_o/C(0)\alpha_2$. Since this lower bound is an increasing function of N, the second part of the theorem follows immediately. ■

V. Effect of Deviation in the Lead Vehicle Trajectory from the Nominal

From the results in the previous section, one might be tempted to conclude that for good tracking of a lead vehicle moving at a constant velocity, $H(s)K(s)$ should be designed to have two integrators. However, as we will see now, doing so may lead to other problems. First of all, a constant velocity may not always be desired. Moreover, even if a constant velocity is desired, it is reasonable to expect that the leader trajectory will deviate from the reference trajectory. In this case, the lead vehicle's trajectory can be modelled as

$$x_o(t) = v_o t + \zeta_o(t),$$

where $\zeta_o(t)$ is the error from the constant-velocity trajectory. We will now show that when $H(s)K(s)$ has two integrators, $||G_{x_o e}||_\infty$ grows without bound as N increases. Note that we use $||.||_2$ to denote the 2-norm of a real or complex vector and $||.||_\infty$ to denote the $\mathcal{H}_\infty$-norm of a transfer function. Since $G_{x_o e}(s)$ is also the transfer function from ζ_o to $\bar{e}$, this means that even if $||\zeta_o||_{\mathcal{L}_2}$ is bounded, $||\bar{e}||_{\mathcal{L}_2}$ will be grow unbounded as N increases. The only situation when good tracking performance is achieved with zero steady state error for all vehicles is when $\zeta_o(t) \equiv 0$, an unlikely scenario.

We first state the main result of this section. This is followed by a technical lemma before the proof of the theorem is presented.

Theorem 3. *Assume $H(s)K(s)$ has two poles at the origin, the closed loop platoon error dynamics under symmetric bidirectional control is stable for arbitrary N. Let $G_{x_oe}(s) \in \mathbb{C}^{N\times 1}$ be the transfer function from lead vehicle position $X_o(s)$ to spacing errors $\bar{E}(s)$ defined in (7). Then, for every $R > 0$, there exists a positive integer $N_o \in \mathbb{N}$ s.t. $||G_{x_oe}||_\infty > R, \ \forall N > N_o$.*

The following lemma will be needed for the proof.

Lemma 3. *Let $C(s)$ be a SISO transfer function that has no poles or zeros at the origin and $C(0) > 0$. Then, $\exists\beta \in (0, +\infty)$ and $\exists N_o \in \mathbb{N}$ such that $\forall N > N_o$,*

$$\sup_\omega \left| \frac{1}{1 - \frac{\lambda_{min}(N)C(j\omega)}{\omega^2}} \right| > \beta N.$$

where $\lambda_{min}(N)$ is the smallest eigenvalue of the matrix $L \in \mathbb{R}^{N\times N}$ defined in (9).

Proof: First we will establish that $|C(j\omega)-C(0)| < \omega\gamma$ for some positive constant γ when ω is small enough. Let $C(s) = N_c(s)/D_c(s)$, where $N_c(s)$ and $D_c(s)$ are coprime polynomials in s (with real coefficients) with degrees m and n, respectively. We write down

$$C(s) = \frac{N(s)}{D(s)} = \frac{z_m s^m + \cdots + z_1 s + z_o}{p_n s^n + \cdots + p_1 s + p_o}$$

where z_o and p_o are non-zero since $C(s)$ does not have poles or zeros at the origin. Expanding the expression for $C(s) - C(0)$ and doing a little algebra, we see that

$$C(s) - C(0) = \frac{s^k Q(s)}{D(s)p_o},$$

where $Q(s)$ is a polynomial in s with a non-zero constant term and $k \geq 1$. Since $Q(s)$ and $D(s)$ both have non-zero constant terms,

$$\lim_{\omega\to 0} \frac{Q(j\omega)}{D(j\omega)p_o} = \frac{Q(0)}{D(0)p_o} \neq 0$$

Therefore,$\exists\omega_o$ s.t. if $|\omega| < \omega_o$, then

$$|\frac{Q(j\omega)}{D(j\omega)p_o}| < |\frac{Q(0)}{D(0)p_o}| + 1 := \gamma.$$

Therefore we get that there exist $\omega_o > 0, \gamma > 0$ and an integer $k \geq 1$ s.t., $\forall |\omega| < \min(1, \omega_o)$,

$$|C(j\omega) - C(0)| \leq |\omega^k|\gamma \leq |\omega|\gamma. \tag{15}$$

Define

$$f(\omega) = \left| \frac{1}{1 - \frac{\lambda_{min}(N)C(j\omega)}{\omega^2}} \right|.$$

Pick N_o such that $\omega^* := \sqrt{\lambda_{min}(N)C(0)} \in (0, \min(1, \omega_o)), \ \forall N > N_o$. Hence,

$$\begin{aligned} f(\omega^*) &= 1/|1 - \frac{C(j\omega^*)}{C(0)}| = \frac{C(0)}{|C(j\omega^*) - C(0)|} \\ &> C(0)/\gamma\omega^*. \end{aligned} \tag{16}$$

The last inequality follows from (15). Substituting the value of ω^*, we get

$$f(\omega^*) > \frac{C(0)^{1/2}}{\gamma\lambda_{min}^{1/2}}, \quad \forall N > N_o. \tag{17}$$

From lemma 2, we know that $1/\lambda_{min}(N) \geq N^2/\alpha_2$. Using this in the inequality (17), we get

$$f(\omega^*) > \beta N, \quad \forall N > N_o \tag{18}$$

where $\beta := (C(0)/\gamma^2\alpha_2)^{1/2}$ is a positive constant. This proves the lemma. ∎

Now we are in a position to prove theorem 3.

Proof: [Proof of Theorem 3] Since $H(s)K(s)$ has two integrators, $H(s)K(s)$ can be written as $C(s)/s^2$. From theorem 1, it follows that $C(s)$ cannot have poles or zeros at 0 and $C(0) > 0$. From (10), we get

$$||G_{x_oe}(s)||_2 = \sqrt{G^*_{x_oe}G_{x_oe}} = \sqrt{\phi_1^T U \Psi^*(s)\Psi(s) U^T \phi_1}.$$

Since the vector $U^T\phi_1$ is the first row of U, using (11), this reduces to

$$||G_{x_oe}(s)||_2 = \left(\sum_{i=1}^N u_{1i}^2 \left| \frac{1}{1 + \lambda_i H(s)K(s)} \right|^2 \right)^{1/2} \tag{19}$$

The $\mathcal{H}_\infty$ norm of the transfer function vector G_{x_oe} is:

$$||G_{x_oe}||_\infty = \sup_\omega ||G_{x_oe}(j\omega)||_2. \tag{20}$$

Thus,

$$||G_{x_oe}||_\infty > \sup_\omega \left(\left| \frac{1}{1 + \lambda_{min} H(j\omega)K(j\omega)} \right| |u_{11}| \right), \tag{21}$$

We can now apply the result established in lemma 3 to claim that $\exists\beta \in (0, +\infty)$ and $\exists N_o \in \mathbb{N}$ such that

$$\sup_\omega \left| \frac{1}{1 + \lambda_{min}(N)H(j\omega)K(j\omega)} \right| > \beta N \quad \forall N > N_o.$$

From lemma 2, we know that $|u_{11}| > 1/\sqrt{N}$. Using these two inequalities in (21), we get

$$||G_{x_oe}||_\infty > \beta N^{1/2} \quad \forall N > N_o.$$

Since this lower bound grows unbounded as N increases, the result follows immediately. ∎

Remark 1. A similar result *does not* hold when $H(s)K(s)$ has only one integrator. In that case we can show that $||G_{x_oe}||_\infty$ can, in fact, be uniformly bounded w.r.t. N. Therefore, if the leader trajectory deviation from a constant-velocity one is $\mathcal{L}_2$-norm bounded, the spacing error resulting from this deviation will be $\mathcal{L}_2$-norm bounded, too, provided the reference trajectory for the lead vehicle is a ramp. However, we have already seen that when $H(s)K(s)$ has one integrator, steady-state spacing errors may be large even without such deviations. We conclude that the question of whether $||G_{x_oe}||_\infty$ can be uniformly bounded when $H(s)K(s)$ has one integrator is not a relevant one, and so refrain from providing a proof of this result.

VI. Disturbance Propagation

To examine the effect of disturbances acting on the vehicles in the spacing errors, we have to look at the transfer function matrix from the disturbances to the spacing errors: $G_{de}(s)$. The question of disturbance propagation was already investigated by Seiler *et. al.* in [5], where it was shown that for the symmetric bidirectional control scheme, it is not possible to design a $K(s)$ to achieve an uniform bound on $||G_{de}||_\infty$ w.r.t. N, when $H(s)$ has two integrators and $K(s)$ has none. It follows from theorem 1 that if $H(s)K(s)$ has three integrators, then the closed loop platoon error dynamics will be unstable for a sufficiently large N. This precludes the possibility of $K(s)$ having an integrator when $H(s)$ has two integrators. Combining this with the result established in [5], we get the following:

Theorem 4. *Assume $H(s)$ has two poles at the origin and the closed loop platoon dynamics is stable with $K(s)$ for arbitrary N under symmetric bidirectional control. Let $G_{de}(s) \in \mathbb{C}^{N\times N}$ be the transfer function matrix from $\bar{D}(s)$ to $\bar{E}(s)$ defined in* (8). *Then, given any $R > 0$, $\exists N_o \in \mathbb{N}$ such that $||G_{de}||_\infty > R$, $\forall N > N_o$.*

This theorem tells us that even if the disturbances acting on the vehicles are $\mathcal{L}_2$-norm bounded, the $\mathcal{L}_2$-norm of the spacing errors due to these disturbances will grow unbounded as N grows. Therefore a symmetric bidirectional control scheme is not scalable with respect to disturbance rejection. This result was established by Seiler *et. al.* in [5] for vehicle models with two integrators, with the assumption that $K(s)$ does not have any integrators. We show that the assumption requiring $K(s)$ not to have an integrator was an artifact of their proof technique.

We have seen in previous section that when $H(s)K(s)$ has two integrators, then at least in one scenario – when the lead vehicle moves at a constant velocity – the closed loop performs well and the spacing errors converge to 0. However, theorem 4 shows that even in that case, if disturbances enter vehicle dynamics, as they invariably will, large spacing errors might result.

VII. Summary and Future Work

It was not known if a symmetric bidirectional control architecture could achieve closed loop stability while keeping the spacing errors bounded, with bounds that are independent of the number of vehicles. We have shown that this is not possible irrespective of the specific controller used. The symmetric bidirectional scheme amplifies spacing errors and disturbances in control signals. The results established in this paper show that the bidirectional control architecture, though attractively decentralized, suffer from fundamental limitations on closed loop performance that cannot be ameliorated by appropriate control design.

There are several related questions that merit study. For instance, we could ask if closed loop performance could be improved by using more than two vehicles' position information, and if so, what are the fundamental limitations in such a scheme? This architecture would now require inter-vehicular communication, but may still be advantageous compared to a predecessor-and-leader following scheme by keeping the distances over which information has to be transmitted, small. Yet another question is, how does the closed loop perform with an *asymmetric* bidirectional architecture, where the control action on a vehicle depends *unequally* on the predecessor and the follower? Future research would attempt to address some of these questions.

References

[1] R. Fax, J.A.; Murray. Information flow and cooperative control of vehicle formations. *IEEE Transactions on Automatic Control*, 49:1465–1476, September 2004.

[2] M. Jovanovic and B. Bamieh. On the ill-posedness of certain vehicular platoon control problems. *submitted to IEEE Transactions on Automatic Control*.

[3] E. Khatir, M.E.; Davison. Decentralized control of a large platoon of vehicles using non-identical controllers. In *Proceedings of the 2004 American Control Conference*, pages 2769–2776, 2004.

[4] C. D. Meyer. *Matrix Analysis and Applied Linear Algebra*. SIAM, 2000.

[5] P. Seiler, A. Pant, and K. Hedrick. Disturbance propagation in vehicle strings. *IEEE Transactions on Automatic Control*, 49:1835–1841, October 2004.

[6] S. S. Stankovic, M. J. Stanojevic, and D. D. Siljak. Decentralized overlapping control of a platoon of vehicles. *IEEE Transactions on Control Systems Technology*, 8: 816–832, September 2000.

[7] D. Swaroop, , and K. Hedrick. String stability of interconnected systems. *IEEE Transactions on Automatic Control*, 41(3):349–356, March 1996.

[8] D. Swaroop, K. Hedrick, C. C. Chien, and P. Ioannou. A comparison of spacing and headway control laws for automatically controlled vehicles. *Vehicle System Dynamics*, 23:597–625, 1994.