

THE MAXIMUM EFFECTIVE RESISTANCE IN 2-DIMENSIONAL AND 3-DIMENSIONAL GRIDS: UPPER BOUNDS FROM WU'S FORMULAE

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Abstract

It is known that the effective resistance between two nodes in an infinite 2-D lattice network of resistors is a logarithmic function of the distance between them [1, 2], and that in a 3-D network, the effective resistance between two nodes stays bounded no matter the distance between the nodes [3]. Recently, exact formulas for the effective resistances between two nodes in *finite* 2-D and 3-D grid network of resistors have been derived by Wu [4]. It follows from the results in [5] that the effective resistance between two fixed nodes in the grids, in the limit when the size of the grids go to infinity, tend to the effective resistance between them in the infinite lattices. This was explicitly verified in [4]. However, it still leaves the question of effective resistance between nodes in the *boundary* of the grids. We use the formulas derived in [4] to obtain upper bounds on the largest effective resistance in 2- and 3-D grid networks, and show that the logarithmic and constant bounds apply even for nodes on the boundary of the finite grids, as long as the grids have “good” aspect ratios.

We find that the largest effective resistance in a $M \times N$ 2-D grid of $n_o := MN$ nodes is $\Theta(\log(n_o))$ subject to the condition that the aspect ratio M/N of the grid is $\Theta(1)$, i.e., there exists positive constants c, \bar{c} such that $c < M/N < \bar{c}$. In case of a $M \times N \times L$ 3-D grid, we find that the largest effective resistance is bounded, independent of the size of the grid as long two dimensions of the grid, say M, N satisfies $M = \Theta(N)$ and the third dimension L satisfies $\Omega(\log(M)) = L = \mathcal{O}(N^2)$.

I. UPPER BOUNDS ON THE MAXIMUM EFFECTIVE RESISTANCE IN 2-D AND 3-D GRIDS

Our starting point is the formula derived by Wu [4] for the effective resistance on a $M \times N$ grid of resistors. Consider a rectangular network of resistors connected in an array of $M \times N$ nodes. A grid with $M = 4$ and $N = 3$ is shown in Figure 1. Number the node coordinates $\{m, n\}$, $0 \leq m \leq M - 1$. The effective resistance between two nodes with coordinates $\{x_1, y_1\}$ and $\{x_2, y_2\}$ is denoted by $R_{M \times N}^{\text{eff}}(\{x_1, y_1\}, \{x_2, y_2\})$. The maximum value of the effective resistance between any two nodes in the grid is denoted by $R_{\text{max}}^{\text{eff}}(M \times N)$. We have the following result:

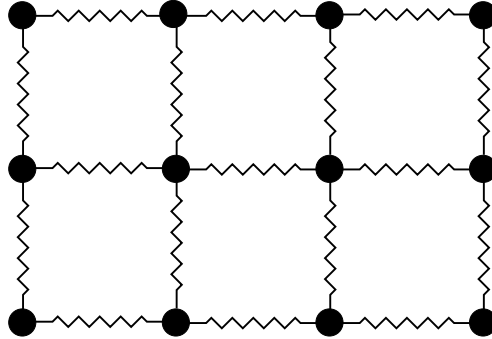


Fig. 1. A 4×3 grid of resistors.

Proposition 1: Consider a $M \times N$ 2-d grid network of 1-Ohm resistors. If there exists positive scalars \underline{c}, \bar{c} independent of M, N such that $\underline{c}N \leq M \leq \bar{c}N$, the maximum effective resistance between any two nodes in the network is

$$R_{\text{max}}^{\text{eff}}(M \times N) = \Theta(\log n_o),$$

where $n_o := MN$ is the total number of nodes in the grid. A non-asymptotic upper bound is

$$R_{\text{max}}^{\text{eff}}(M \times N) \leq \frac{M}{N} + \frac{N}{M} + 3 \log(MN) + 6. \quad \square$$

Note that the notation $g(x) = \mathcal{O}(p(x))$ means that there exist positive constants x_o, a such that $g(x) \leq ap(x)$ for all $x > x_o$. Similarly, $g(x) = \Omega(p(x))$ means there exist a positive constants x_o, b such that $g(x) \geq bp(x)$ for all $x > x_o$. The notation $g(x) = \Theta(p(x))$ means that both $g(x) = \mathcal{O}(p(x))$ and $g(x) = \Omega(p(x))$ are true.

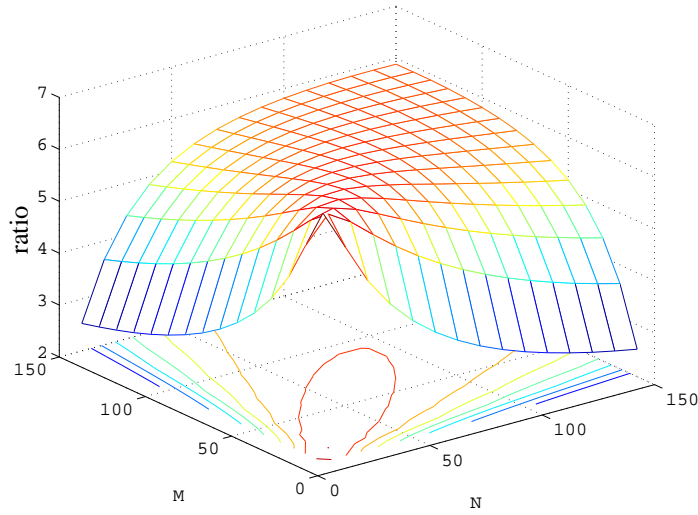


Fig. 2. Numerical comparison of the upper bound on the maximum effective resistance to the exact value for 2D grids. The plot shows the ratio between the upper bound stated in Proposition 1 on $R_{\max}^{\text{eff}}(M \times N)$ (maximum effective resistance between two nodes in the $M \times N$ grid of unit resistors) and its exact value computed from the formula in [4], for a range of values of M and N . The comparison shows that the upper bound derived overpredicts the maximum effective resistance at most by a factor of 7.

Now consider a $M \times N \times L$ 3-d grid of resistors, with one Ohm resistor on every edge. The result on the maximum effective resistance between any two nodes in the network, denoted by $R_{\max}^{\text{eff}}(M \times N \times L)$ is stated in the next result.

Proposition 2: The maximum effective resistance between any two nodes in a $M \times N \times L$ 3-d grid of 1-Ohm resistors is bounded by a constant independent of M, N, L when any two dimensions of the 3-d grid have the same order and the third dimension is at least logarithmic but no more than quadratic in the other two dimensions. More precisely, if there exists positive scalars $\underline{c}, \bar{c}, \underline{d}, \bar{d}$ such that $\underline{c}N \leq M \leq \bar{c}N$ and $\underline{d} \log N \leq L \leq \bar{d}N^2$, then

$$R_{\max}^{\text{eff}}(M \times N \times L) = \Theta(1). \quad \square$$

A non-asymptotic upper bound on the maximum effective resistance in a 3D grid is

$$R_{\max}^{\text{eff}}(M \times N \times L) \leq 2 \frac{M^2 + N^2 + L^2}{MNL} + 4 \frac{1}{M} \log NL + 4 \frac{1}{N} \log LM + 4 \frac{1}{L} \log MN + c,$$

where $c < 50$. □

II. NUMERICAL COMPARISON

Numerical comparison of the derived upper bound and the true value of the maximum effective resistance in a 2-D grid is shown in Figure 2, which shows the ratio between the non-asymptotic upper bound stated in Proposition 1 on $R_{\max}^{\text{eff}}(M \times N)$ and its value computed from the formula in [4], for a range of values of M and N . The comparison shows that the upper bound derived overpredicts the maximum effective resistance by a factor of 7.

Numerical comparison of the derived bound and the true value for a 3-D grid is shown in Figure 3, which shows the ratio between the non-asymptotic upper bound stated in Proposition 2 on $R_{\max}^{\text{eff}}(M \times N \times L)$ and its true value computed from the formula in [4], for a range of values of M, N and L , as a color intensity plot. The comparison shows that the upper bound for the 3-D case is quite conservative – it overpredicts the maximum effective resistance by a factor that can be as large as 42.

III. PROOFS

Proof of Proposition 1. The effective resistance between the nodes $\{x_1, y_1\}$ and $\{x_2, y_2\}$ is given by the following formula [4]:

$$R_{\{M \times N\}}^{\text{eff}}(\{x_1, y_1\}, \{x_2, y_2\}) = \frac{r}{N} |x_1 - x_2| + \frac{s}{M} |y_1 - y_2| + \frac{2}{MN} \times \sum_{n=1}^{N-1} \sum_{m=1}^{M-1} \frac{[\cos(x_1 + \frac{1}{2})\theta_m \cos(y_1 + \frac{1}{2})\phi_n - \cos(x_2 + \frac{1}{2})\theta_m \cos(y_2 + \frac{1}{2})\phi_n]^2}{(1 - \cos \theta_m) + (1 - \cos \phi_n)}, \quad (1)$$

where

$$\theta_m = \frac{m\pi}{M}, \phi_n = \frac{n\pi}{N}, \quad (2)$$

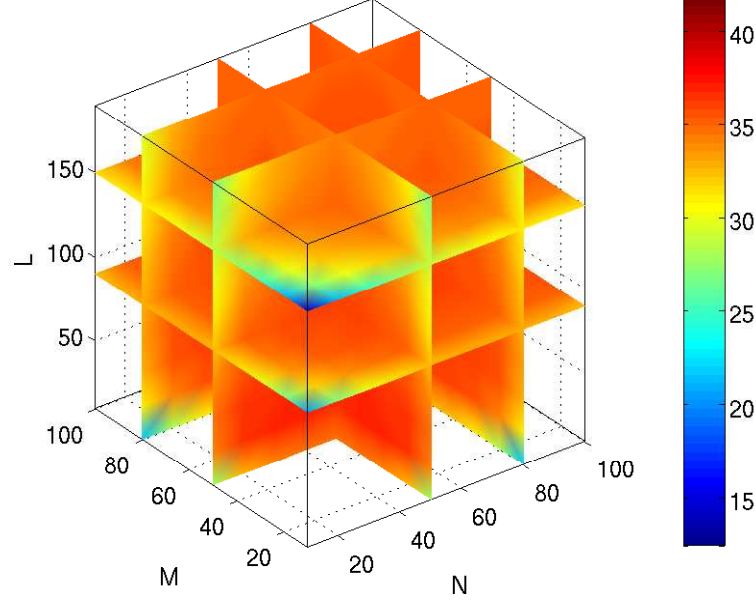


Fig. 3. Numerical comparison of the upper bound on the maximum effective resistance to the exact value for 3D grids. The color intensity is the ratio between the upper bound stated in Proposition 2 on $R_{\max}^{\text{eff}}(M \times N \times L)$ (maximum effective resistance between two nodes in the $M \times N \times L$ grid of unit resistors) and its exact value computed from the formula in [4], for a range of values of M, N and N . The comparison shows that the upper bound derived is quite conservative – it overpredicts the maximum effective resistance at most by a factor of 42.

so that both θ_m and ϕ_n lie in $(0, \pi)$ for all allowable values of m and n . Clearly,

$$R_{\{M \times N\}}^{\text{eff}}(\{x_1, y_1\}, \{x_2, y_2\}) \leq \frac{r}{N}|x_1 - x_2| + \frac{s}{M}|y_1 - y_2| + \frac{2}{MN} \times \sum_{n=1}^{N-1} \sum_{m=1}^{M-1} \frac{2^2}{2 - \cos \theta_m - \cos \phi_n}$$

The right hand side is maximized by choosing $|x_1 - x_2| = M - 1$ and $|y_1 - y_2| = N - 1$, which gives us

$$R_{\max}^{\text{eff}}(M \times N) \leq \frac{M-1}{N} + \frac{N-1}{M} + \frac{8}{MN} \sum_{n=1}^{N-1} \sum_{m=1}^{M-1} \frac{1}{2 - \cos \theta_m - \cos \phi_n} \quad (3)$$

Now define

$$\begin{aligned} \alpha_n &:= 2 - \cos \phi_n, & \Rightarrow \alpha_n \in (1, 3), n = 1, \dots, N-1 \\ f_{m,n} &:= \frac{1}{\alpha_n - \cos \theta_m}, \end{aligned}$$

so that

$$T_{MN} := \frac{8}{MN} \sum_{n=1}^{N-1} \sum_{m=1}^{M-1} \frac{1}{2 - \cos \theta_m - \cos \phi_n} = \frac{8}{MN} \sum_{n=1}^{N-1} \sum_{m=1}^{M-1} f_{m,n}. \quad (4)$$

Let

$$f(\theta) := \frac{1}{\alpha_n - \cos \theta}, \quad \theta \in (0, \pi)$$

which is a decreasing function of $\theta \in (0, \pi)$. This can be verified in a straightforward manner by computing the derivative of $f(\theta)$ w.r.t θ . Therefore,

$$\begin{aligned} \Delta\theta \sum_{m=1}^{M-1} f_{m,n} &\leq \int_{0+}^{\pi} f(\theta) d\theta, & \text{where } \Delta\theta &:= \frac{\pi}{M}. \\ &= \int_{0+}^{\pi} \frac{1}{\alpha_n - \cos \theta} d\theta \end{aligned} \quad (5)$$

Since $\alpha_n > 1$, from [6]

$$\begin{aligned} \int \frac{1}{\alpha - \cos \theta} d\theta &= \frac{2}{\sqrt{\alpha^2 - 1}} \arctan \left[\frac{\sqrt{\alpha^2 - 1} \tan \frac{\theta}{2}}{\alpha - 1} \right] + \text{const.} \\ \Rightarrow \int_{0^+}^{\pi} \frac{1}{\alpha_n - \cos \theta} d\theta &= \frac{2}{\sqrt{\alpha_n^2 - 1}} [\arctan(+\infty) - \arctan(0^+)] = \frac{\pi}{\sqrt{\alpha_n^2 - 1}}, \end{aligned}$$

and therefore, from (5)

$$\frac{1}{M} \sum_{m=1}^{M-1} f_{m,n} \leq \frac{1}{\sqrt{\alpha_n^2 - 1}}$$

Now, $\alpha_n^2 - 1 = (\alpha_n + 1)(\alpha_n - 1) > 2(1 - \cos \frac{1}{2}\phi_n) = 4 \sin^2(\frac{1}{2}\phi_n)$. Moreover, it can be shown from the series expansion of $\sin x$ around 0 that

$$2 \sin x \geq x, \forall x \in (0, \frac{\pi}{2}). \quad (6)$$

Therefore, $\sqrt{\alpha_n^2 - 1} > \frac{1}{2}\phi_n$. As a result,

$$T_{MN} < \frac{16}{N} \sum_{n=1}^{N-1} \frac{1}{\phi_n} \quad (7)$$

Since $1/\phi$ is positive and a decreasing function of ϕ in the interval $(0, \pi)$,

$$\begin{aligned} \Delta\phi \sum_{n=1}^N \frac{1}{\phi_n} &= \Delta\phi \frac{1}{\phi_1} + \Delta\phi \sum_{n=2}^{N-1} \frac{1}{\phi_n}, \quad \text{where } \Delta\phi := \frac{\pi}{N} = \phi_1 \\ &\leq 1 + \int_{\phi_1}^{\pi} \frac{1}{\phi} d\phi = 1 + \log N \end{aligned}$$

from which it follows that

$$T_{MN} \leq \frac{16}{\pi}(1 + \log N)$$

Putting this bound in (3), we get

$$R_{\max}^{\text{eff}} \leq \frac{M}{N} + \frac{N}{M} + \frac{16}{\pi}(1 + \log N). \quad (8)$$

By repeating the steps once again but summing first over n and then over m , we also get

$$R_{\max}^{\text{eff}} \leq \frac{M}{N} + \frac{N}{M} + \frac{16}{\pi}(1 + \log M). \quad (9)$$

Combining (8) and (9), we get

$$R_{\max}^{\text{eff}} \leq \frac{M^2 + N^2}{MN} + \frac{8}{\pi} \log MN + \frac{16}{\pi} \quad (10)$$

Since $n_o = MN$ and $\frac{8}{\pi} < 3$, we get the second statement of the proposition. When there exists positive scalars \underline{c}, \bar{c} such that $\underline{c}N \leq M \leq \bar{c}N$, then we get

$$R_{\max}^{\text{eff}} \leq \bar{c} + \frac{1}{\underline{c}} + \frac{16}{\pi} + \frac{8}{\pi} \log n_o = \mathcal{O}(\log n_o). \quad (11)$$

As a result, it follows that in a rectangular grid of unit resistors with n_o nodes arranged in such a way that the aspect ratio of the grid is $\Theta(1)$, the maximum effective resistance is $\mathcal{O}(\log(n_o))$.

Effective resistances obey Rayleigh's monotonicity law [7], which says that the effective resistance between any two nodes can only decrease if one introduces new nodes and branches into an electrical network. It follows that the effective resistance between every pair of nodes in a finite grid is larger than that between them in the corresponding infinite grid. The effective resistance between the nodes at $(0, 0)$ and (x, y) in an infinite 2-d network with 1-Ohm resistors on every branch is [3]

$$R_{\infty 2\text{-d lattice}}^{\text{eff}}(\{0, 0\}, \{x, y\}) = \frac{1}{\pi} \left(\frac{1}{2} \log(x^2 + y^2) + \gamma + \frac{\log 8}{2} \right),$$

where $\gamma = 0.577\dots$. Therefore,

$$R_{\max}^{\text{eff}}(M \times N) \geq \frac{1}{\pi} \left(\frac{1}{2} \log(M^2 + N^2) + \gamma + \frac{\log 8}{2} \right).$$

Since $M^2 + N^2 \geq 2MN = 2n_o$, the right hand side is $\Omega(\log n_o)$. Therefore, both the upper and the lower bounds on the maximum effective resistance are logarithmic, and so

$$\max R_{\{M \times N\}}^{\text{eff}} = \Theta(\log n_o), \quad \text{when } M = \Theta(N).$$

This concludes the proof. ■

Proof of Proposition 2. Again we start from the formula derived by Wu [4] for the effective resistance between two nodes in a $M \times N \times L$ grid of resistors, with one Ohm resistor on every edge. The effective resistance between two nodes with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the following formula [4]:

$$\begin{aligned} R_{\{M \times N \times L\}}^{\text{eff}}(\{x_1, y_1, z_1\}, \{x_2, y_2, z_2\}) &= \frac{1}{L} R_{\{M \times N\}}^{\text{eff}}(\{x_1, y_1\}, \{x_2, y_2\}) + \frac{1}{M} R_{\{N \times L\}}^{\text{eff}}(\{y_1, z_1\}, \{y_2, z_2\}) \\ &+ \frac{1}{N} R_{\{L \times M\}}^{\text{eff}}(\{z_1, x_1\}, \{z_2, x_2\}) - \frac{1}{MN} |x_1 - x_2| - \frac{1}{NL} |y_1 - y_2| - \frac{1}{LM} |z_2 - z_1| \\ &+ T_{MNL}, \end{aligned}$$

where

$$T_{MNL} = \frac{4}{MNL} \times \sum_{l=1}^{L-1} \sum_{n=1}^{N-1} \sum_{m=1}^{M-1} \frac{[\cos(x_1 + \frac{1}{2})\theta_m \cos(y_1 + \frac{1}{2})\phi_n \cos(z_1 + \frac{1}{2})\eta_l - \cos(x_2 + \frac{1}{2})\theta_m \cos(y_2 + \frac{1}{2})\phi_n \cos(z_2 + \frac{1}{2})\eta_l]^2}{(1 - \cos \theta_m) + (1 - \cos \phi_n) + (1 - \cos \eta_l)},$$

and θ_m, ϕ_n are defined in (2) and

$$\eta_l = \frac{l\pi}{L} \quad \Rightarrow \eta_l \in (0, \pi), l = 1, \dots, L-1.$$

It follows from the above that

$$R_{\text{max}}^{\text{eff}}(M \times N \times L) \leq \frac{1}{L} R_{\text{max}}^{\text{eff}}(M \times N) + \frac{1}{M} R_{\text{max}}^{\text{eff}}(N \times L) + \frac{1}{N} R_{\text{max}}^{\text{eff}}(L \times M) + T_{MNL}. \quad (12)$$

First we show that the term T_{MNL} is bounded. Clearly,

$$T_{MNL} \leq \frac{4}{MNL} \times \sum_{l=1}^{L-1} \sum_{n=1}^{N-1} \sum_{m=1}^{M-1} \frac{2^2}{(3 - \cos \theta_m - \cos \phi_n - \cos \eta_l)}, \quad (13)$$

We proceed as in the 2-D case, by defining

$$\alpha_{n,l} = 3 - \cos \phi_n - \cos \eta_l, \quad \Rightarrow \alpha_{n,l} \in (1, 5) \quad n = 1, \dots, N-1, \quad l = 1, \dots, L-1. \quad (14)$$

and repeating the steps as in the 2-D case, arriving at

$$T_{MNL} \leq \frac{16}{NL} \sum_{l=1}^{L-1} \sum_{n=1}^{N-1} \frac{1}{\sqrt{\alpha_{n,l}^2 - 1}}$$

Define

$$\beta_l := 1 - \cos \eta_l = 2 \sin^2\left(\frac{\eta_l}{2}\right) \Rightarrow \beta_l \in (0, 2) \quad (15)$$

so that $\alpha_{n,l}^2 - 1 = (\alpha_{n,l} + 1)(\alpha_{n,l} - 1) > (1 - \cos \phi_n + 1 - \cos \eta_l) = \beta_l + 2 \sin^2 \frac{1}{2} \phi_n$. Moreover, applying (6) for $\frac{1}{2} \phi_n \in (0, \frac{\pi}{2})$, we get $\sin \frac{1}{2} \phi_n \geq \frac{1}{4} \phi_n$, from which we finally get

$$\frac{1}{\sqrt{\alpha_{n,l}^2 - 1}} \leq \frac{1}{\sqrt{2(\beta_l + \frac{1}{8} \phi_n^2)}}$$

Since $\frac{1}{\sqrt{a+x^2}}$ is a decreasing function of x , we get

$$\Delta \phi \sum_{n=1}^{N-1} \frac{1}{\sqrt{\alpha_{n,l}^2 - 1}} \leq 2 \int_0^\pi \frac{1}{\sqrt{8\beta_l + \phi^2}} d\phi = 2 \log \left| \frac{\pi + \sqrt{8\beta_l + \pi^2}}{\sqrt{8\beta_l}} \right| \leq \frac{1}{10} - \log \beta_l$$

where the last inequality results from using $\beta_l < 2$ for all allowable l . Applying (6) once again, we get $-\log \beta_l \leq 3 \log 2 - 2 \log \eta_l$. Combining all these, we get

$$\frac{1}{N} \sum_{n=1}^{N-1} \frac{1}{\sqrt{\alpha_{n,l}^2 - 1}} \leq 1 - \frac{2}{\pi} \log \eta_l$$

Therefore,

$$T_{MNL} \leq 16 + \frac{32}{\pi L} \sum_{l=1}^{L-1} (-\log \eta_l).$$

Since $-\log(x)$ is a decreasing function of x , we get

$$\begin{aligned} \sum_{l=1}^{L-1} (-\log \eta_l)(\Delta\eta) &\leq (-\log \eta_1)\Delta\eta + \int_{\eta_1}^{\pi} -\log \eta \, d\eta; \quad \text{where } \Delta\eta := \frac{\pi}{L} \\ &= \log\left(\frac{L}{\pi}\right)\Delta\eta - (x \log x - x) \Big|_{\eta_1}^{\pi} \end{aligned}$$

Pluggin in $\eta_1 = \frac{\pi}{L}$ in the above, we obtain

$$\begin{aligned} -\frac{1}{L} \sum_{l=1}^{L-1} \log \eta_l &\leq \frac{1}{L} \log \frac{L}{\pi} + (\log \pi - 1)\left(1 - \frac{1}{L}\right) + \frac{1}{L} \log L < 3 \quad \forall L \geq 1, \\ \Rightarrow T_{MNL} &\leq 50. \end{aligned}$$

On applying this bound and results from Proposition 1 in (12), we get

$$R_{\max}^{\text{eff}}(M \times N \times L) \leq 2 \frac{M^2 + N^2 + L^2}{MNL} + 4 \frac{1}{M} \log NL + 4 \frac{1}{N} \log LM + 4 \frac{1}{L} \log MN + 50 \quad (16)$$

It is straightforard to show that when M, N, L are related by the aspect ratios stated in the proposition, the right hand side of (16) is bounded by a constant independent of M, N, L even when $M, N, L \rightarrow \infty$. This concludes the proof. ■

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